Soft Graviton Theorem in Generic Quantum Theory of Gravity

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Tel Aviv, June 2017

What is soft graviton theorem?

Take a general coordinate invariant quantum field theory of gravity coupled to matter fields

Consider an S-matrix element involving

– arbitrary number of external particles of finite energy

– one external graviton carrying small momentum k.

Soft graviton theorem: Expansion of this amplitude in power series in k in terms of the amplitude without the graviton.

There are many results.

1. **General results at leading order in k** — Weinberg; . . .

2. **General subleading results in four dimensions via BMS** — Strominger; Strominger, Zhiboedov; Campiglia, Laddha; . . .

3. **Results in specific theories in general dimensions**

White; Cachazo, Strominger; Bern, Davies, Di Vecchia, Nohle; Elvang, Jones, Naculich; . . .
Bianchi, Guerrieri; Di Vecchia, Marotta, Mojaza; . . .

Our goal: Study soft graviton amplitudes in generic theories in generic number of dimensions for arbitrary mass and spin of external states

– including string theory

Limitations: We have to work in non-compact space-time dimensions D $\geq$ 5 in order to avoid infrared divergences.

In D=4 the S-matrix elements themselves are infrared divergent, introducing additional subtleties.

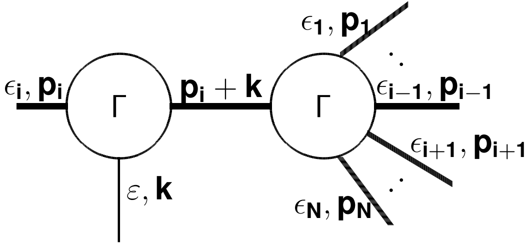
Bern, Davies, Nohle

In D=4, our analysis will apply to only tree amplitudes.

A.S. arXiv:1702.03934, 1703.00024

A. Laddha, A.S., arXiv:1706.00759

We have to consider two types of Feynman diagrams

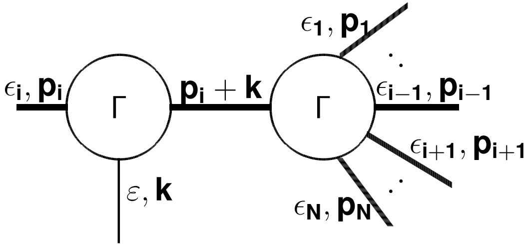


Γ: amputated Green's function

Internal lines: Full renormalized propagators

ε, k: polarization, momentum of soft graviton

ϵ_i, p_i: polarization, momentum of finite energy external particles.



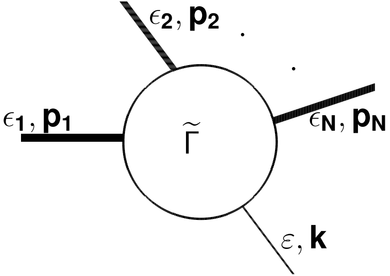
The internal line carrying momentum $p_i + k$ has denominator factor

$$\{(p_i + k)^2 + M_i^2\}^{-1} = (2p_i \cdot k)^{-1}$$

using $p_i^2 + M_i^2 = 0$, $k^2 = 0$.

$\Rightarrow$ this starts contributing at the leading order.

Second class of diagrams



$\widetilde{\Gamma}$: **Amputated amplitudes in which the external soft graviton does not get attached to an external line**

– has no pole as k $\to$ 0

$\Rightarrow$ the contribution from this diagram begins at the subleading order.

Strategy for computation

1. Consider the gauge invariant one particle irreducible (1PI) effective action of the theory

2. Expand the action in powers of all fields, including the metric fluctuations, around the extremum of the action

3. Add manifestly Lorentz invariant gauge fixing terms.

4. This action is used to compute vertices and propagators of finite energy external states but not of soft external gravitons.

5. To calculate the coupling of the soft graviton $S_{\mu\nu}$ to the rest of the fields, we <u>covariantize</u> the gauge fixed action.

a. Replace the background metric $\eta_{\mu\nu}$ by $\eta_{\mu\nu} + 2\,S_{\mu\nu}$

b. Replace all derivatives by covariant derivatives computed with the metric $\eta_{\mu\nu} + 2\,S_{\mu\nu}$

This misses terms involving Riemann tensor computed from the metric $\eta_{\mu\nu} + 2\,S_{\mu\nu}$ but that contains two derivatives and hence is sub-subleading.

1. We choose

$$\mathbf{S}_{\mu\nu} = \varepsilon_{\mu\nu}\mathbf{e}^{\mathbf{ik\cdot x}}, \quad \varepsilon_{\mu\nu} = \varepsilon_{\nu\mu}, \quad \varepsilon^{\mu}{}_{\mu} = \mathbf{k}^{\mu}\varepsilon_{\mu\nu} = \mathbf{0}$$

All indices raised and lowered by η.

2. All fields representing finite energy external states are taken to carry tangent space Lorentz indices

– allows us to give uniform treatment to fermions and bosons.

3. To first order in $\mathbf{S}_{\mu\nu}$, we take the vielbeins to be

$$\mathbf{e}_{\mu}{}^{\mathbf{a}} = \delta_{\mu}{}^{\mathbf{a}} + \mathbf{S}_{\mu}{}^{\mathbf{a}}, \quad \mathbf{E}_{\mathbf{a}}{}^{\mu} = \delta_{\mathbf{a}}{}^{\mu} - \mathbf{S}_{\mathbf{a}}{}^{\mu}, \quad \mathbf{g}^{\mu\nu} = \eta^{\mu\nu} - \mathbf{2}\,\mathbf{S}^{\mu\nu}$$

Covariantization: Acting on a field ϕ_α:

$$\partial_{a_1} \cdots \partial_{a_n} \Rightarrow E_{a_1}{}^{\mu_1} \cdots E_{a_n}{}^{\mu_n} D_{\mu_1} \cdots D_{\mu_n}, \quad E_a{}^\mu \equiv (\delta_a{}^\mu - S_a{}^\mu)$$

$$D_\mu \phi_\alpha = \partial_\mu \phi_\alpha + \frac{1}{2} \omega_\mu^{ab} (J_{ab})_\alpha{}^\gamma \phi_\gamma$$

$$D_{(\mu} D_{\nu)} \phi_\alpha = \partial_\mu \partial_\nu \phi_\alpha + \cdots + \frac{1}{2} \partial_{(\mu} \omega_{\nu)}^{ab} (J_{ab})_\alpha{}^\gamma \phi_\gamma + \left\{ \begin{matrix} \rho \\ \mu\ \nu \end{matrix} \right\} \partial_\rho \phi_\alpha$$

etc.

$$\omega_\mu^{ab} = \partial^b S_\mu{}^a - \partial^a S_\mu{}^b, \quad S_{\mu\nu} = \varepsilon_{\mu\nu} e^{ik\cdot x}$$

$$\left\{ \begin{matrix} \rho \\ \mu\ \nu \end{matrix} \right\} = \frac{1}{2} \left[\partial_\mu S_\nu{}^\rho + \partial_\nu S_\mu{}^\rho - \partial^\rho S_{\mu\nu} \right]$$

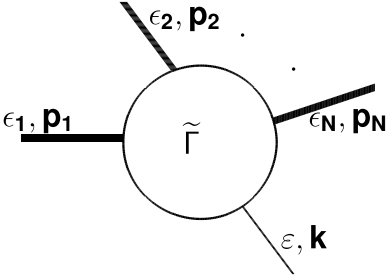
Consider a functional

$$\int \mathbf{d^D p_1} \cdots \mathbf{d^D p_N}\, \phi_{\alpha_1}(\mathbf{p_1}) \cdots \phi_{\alpha_N}(\mathbf{p_N})$$
$$\delta^{(D)}(\mathbf{p_1} + \cdots \mathbf{p_N})\, \mathbf{F}^{\alpha_1 \cdots \alpha_N}(\mathbf{p_1}, \ldots \mathbf{p_N})$$

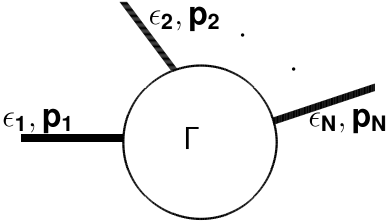
Covariantization produces an additional term

$$\int \mathbf{d^D p_1} \cdots \mathbf{d^D p_N}\, \phi_{\alpha_1}(\mathbf{p_1}) \cdots \phi_{\alpha_N}(\mathbf{p_N})\, \delta^{(D)}(\mathbf{p_1} + \cdots \mathbf{p_N} + \mathbf{k})$$
$$\sum_{i=1}^{N} \Big[-\delta_{\beta_i}^{\ \alpha_i} \varepsilon_\mu^{\ \nu} \mathbf{p}_{i\nu} \frac{\partial}{\partial \mathbf{p}_{i\mu}} + \mathbf{k}^b \varepsilon_\mu^{\ a} (\mathbf{J}_{ab})_{\beta_i}^{\ \alpha_i} \frac{\partial}{\partial \mathbf{p}_{i\mu}}$$
$$-\frac{1}{2} \delta_{\beta_i}^{\ \alpha_i} \left\{ \mathbf{k}_\mu \varepsilon_\nu^{\ \rho} + \mathbf{k}_\nu \varepsilon_\mu^{\ \rho} - \mathbf{k}^\rho \varepsilon_{\mu\nu} \right\} \mathbf{p}_{i\rho} \frac{\partial^2}{\partial \mathbf{p}_{i\mu} \partial \mathbf{p}_{i\nu}} \Big]$$
$$\mathbf{F}^{\alpha_1 \cdots \alpha_{i-1} \beta_i \alpha_{i+1} \cdots \alpha_N}(\mathbf{p_1}, \ldots, \mathbf{p_N}) + \mathcal{O}(\mathbf{k}^\mu \mathbf{k}^\nu)\,.$$

Now consider

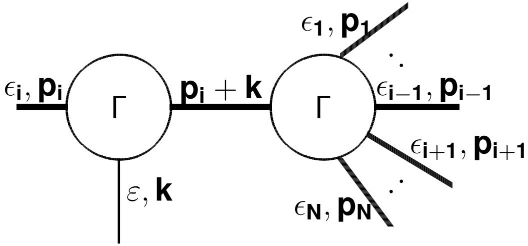


1. Take the amplitude without soft graviton.



2. Covariantize it to order k^μ

Next consider



Need to focus on the three point coupling computed from the 1PI action.

Begin with two point function without the soft graviton and covariantize it to order $k^\mu k^\nu$.

$$\mathbf{S}^{(2)} = \frac{1}{2}\int \frac{d^D q_1}{(2\pi)^D}\,\frac{d^D q_2}{(2\pi)^D}\,\phi_\alpha(q_1)\mathcal{K}^{\alpha\beta}(q_2)\phi_\beta(q_2)\,(2\pi)^D\delta^{(D)}(q_1+q_2)$$

$\{\phi_\alpha\}$: **set of all the fields**

$\mathcal{K}^{\alpha\beta}(q)$: **Kinetic operator, chosen to satisfy**

$$\mathcal{K}^{\alpha\beta}(q) = \mathcal{K}^{\beta\alpha}(-q)$$

Covariantization $\Rightarrow$ coupling of ϕ_α to soft graviton

$$\begin{aligned} S^{(3)} \;=\;& \frac{1}{2}\int \frac{d^D q_1}{(2\pi)^D}\,\frac{d^D q_2}{(2\pi)^D}\,(2\pi)^D\delta^{(D)}(q_1+q_2+k) \\ & \times\phi_\alpha(q_1)\Big[-\varepsilon_{\mu\nu}q_2^\nu\frac{\partial}{\partial q_{2\mu}}\mathcal{K}^{\alpha\beta}(q_2) - k_a\,\varepsilon_{b\mu}\frac{\partial}{\partial q_{2\mu}}\mathcal{K}^{\alpha\gamma}(q_2)\,\left(J^{ab}\right)_\gamma^{\ \beta} \\ & -\frac{1}{2}\delta_\beta^\alpha\left\{k_\mu\varepsilon_\nu^{\ \rho}+k_\nu\varepsilon_\mu^{\ \rho}-k^\rho\varepsilon_{\mu\nu}\right\}\,q_{2\rho}\,\frac{\partial^2}{\partial q_{2\mu}\partial q_{2\nu}}\Big]\,\phi_\beta(q_2)+\mathcal{O}(k^\mu k^\nu)\end{aligned}$$

– determines the coupling of the soft graviton to the finite energy particles

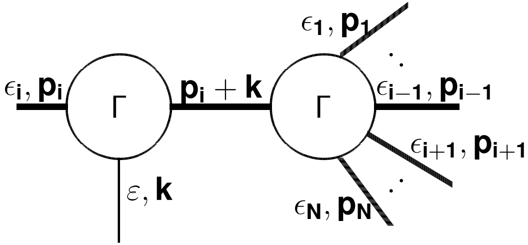
We also need the propagator of the particle carrying momentum $\mathbf{p_i} + \mathbf{k}$

– given by $\quad \mathbf{i}\,\mathcal{K}^{-1}_{\alpha\beta}(\mathbf{p_i}+\mathbf{k}) \equiv (\mathbf{2p_i}\cdot\mathbf{k})^{-1}\mathcal{N}^{\mathbf{i}}_{\alpha\beta}(\mathbf{p_i}+\mathbf{k})$

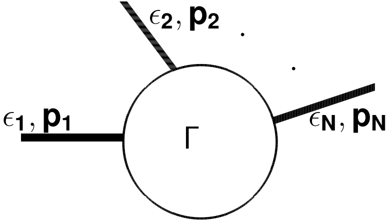
The final ingredient is the on-shell condition

$$\epsilon_{\mathbf{i},\alpha}\,\mathcal{K}^{\alpha\beta}(\mathbf{p_i}) = \mathbf{0}$$

We now substitute and compute



Final result is stated in terms of the amplitude without the soft graviton



Call this

$$\epsilon_{1,\alpha_1}(p_1)\ldots\epsilon_{N,\alpha_N}(p_N)\,\Gamma^{\alpha_1\ldots\alpha_N}(p_1,\ldots,p_N)$$

α_i: tangent space tensor / spinor indices labelling all the fields of the theory

$$\Gamma^{\alpha_i}_{(i)}(p_i)\equiv\prod_{j\neq i}\epsilon_{j,\alpha_j}(p_j)\,\Gamma^{\alpha_1\ldots\alpha_N}(p_1,\ldots,p_N)$$

Final result for the soft graviton amplitude to subleading order:

$$\begin{aligned}
&\Gamma(\varepsilon, \mathbf{k}; \epsilon_1, \mathbf{p_1}; \ldots; \epsilon_N, \mathbf{p_N}) \\
= &\sum_{i=1}^{N} (\mathbf{p_i} \cdot \mathbf{k})^{-1} \, \varepsilon_{\mu\nu} \, \mathbf{p_i^\mu} \, \mathbf{p_i^\nu} \epsilon_{i,\alpha} \Gamma^{\alpha}_{(i)}(\mathbf{p_i}) \\
+ &\sum_{i=1}^{N} (\mathbf{p_i} \cdot \mathbf{k})^{-1} \, \epsilon_{i,\alpha} \, \varepsilon_{\mu\nu} \, \mathbf{p_i^\mu} \, \mathbf{k_\rho} \left(\mathbf{p_i^\nu} \frac{\partial}{\partial \mathbf{p_{i\rho}}} - \mathbf{p_i^\rho} \frac{\partial}{\partial \mathbf{p_{i\nu}}} \right) \Gamma^{\alpha}_{(i)}(\mathbf{p_i}) \\
+ &\sum_{i=1}^{N} (\mathbf{p_i} \cdot \mathbf{k})^{-1} \, \epsilon_{i,\alpha} \, \varepsilon_{\mu b} \, \mathbf{p_i^\mu} \, \mathbf{k_a} \, (\mathbf{J^{ab}})_{\gamma}^{\ \alpha} \, \Gamma^{\gamma}_{(i)}(\mathbf{p_i}) \,,
\end{aligned}$$

This is the subleading soft graviton theorem

– agrees with all known results in field theory / string theory

The sub-subleading amplitude has a universal term and a non-universal term.

Universal term:

$$\frac{1}{2}\sum_{i=1}^{N}(p_i\cdot k)^{-1}\epsilon_{i,\alpha}\varepsilon_{ac}k_bk_d\left[\left\{p_i^b\frac{\partial}{\partial p_{ia}}-p_i^a\frac{\partial}{\partial p_{ib}}\right\}\delta_\beta{}^\alpha+(J^{ab})_\beta{}^\alpha\right]$$
$$\left[\left\{p_i^d\frac{\partial}{\partial p_{ic}}-p_i^c\frac{\partial}{\partial p_{id}}\right\}\delta_\gamma{}^\beta+(J^{cd})_\gamma{}^\beta\right]\Gamma^\gamma_{(i)}(p_i)$$

Non-universal term

$$\frac{1}{2}\left(\varepsilon^{\mu\nu}k^\rho k^\sigma - \varepsilon^{\mu\sigma}k^\nu k^\rho - \varepsilon^{\nu\rho}k^\sigma k^\mu + \varepsilon^{\rho\sigma}k^\mu k^\nu\right)$$
$$\sum_{i=1}^{N}(p_i\cdot k)^{-1}\,\epsilon_{i,\alpha}\,M^\alpha_{\beta;\mu\rho\nu\sigma}(-p_i)\,\Gamma^\beta_{(i)}(p_i)$$

$$\mathbf{M}^{\alpha}_{\delta;\mu\rho\nu\sigma}(-\mathbf{p_i})$$

$$= \Big\{ \frac{1}{3}\,\mathbf{i}\,\mathbf{p}^{\nu}\, \frac{\partial\mathcal{K}^{\alpha\beta}(-\mathbf{p_i})}{\partial \mathbf{p_{i}}_{\mu}}\, \frac{\partial^2 \mathcal{N}^{\mathbf{i}}_{\beta\delta}(-\mathbf{p_i})}{\partial \mathbf{p_i}_{\rho}\partial \mathbf{p_i}_{\sigma}} - \frac{1}{6}\,\mathbf{i}\, \frac{\partial^2\mathcal{K}^{\alpha\beta}(-\mathbf{p_i})}{\partial \mathbf{p_i}_{\mu}\partial \mathbf{p_i}_{\nu}}\, \mathbf{p_i}^{\rho}\, \frac{\partial \mathcal{N}^{\mathbf{i}}_{\beta\delta}(-\mathbf{p_i})}{\partial \mathbf{p_i}_{\sigma}}$$

$$+\frac{\mathbf{i}}{4}\,\frac{\partial\mathcal{K}^{\alpha\gamma}(-\mathbf{p_i})}{\partial \mathbf{p_i}_{\mu}}\,\frac{\partial \mathcal{N}^{\mathbf{i}}_{\gamma\beta}(-\mathbf{p_i})}{\partial \mathbf{p_i}_{\rho}}\,(\mathbf{J}^{\nu\sigma})_{\delta}^{\ \beta} - \frac{1}{4}(\mathbf{J}^{\mu\rho})_{\beta}^{\ \alpha}(\mathbf{J}^{\nu\sigma})_{\delta}^{\ \beta}$$

$$+\mathbf{i}\,\mathcal{B}^{\alpha\beta;\mu\rho\nu\sigma}(-\mathbf{p_i})\,\mathcal{N}^{\mathbf{i}}_{\beta\delta}(-\mathbf{p_i})\Big\}$$

$$\mathcal{N}^{\mathbf{i}}(\mathbf{p}) = \mathbf{i}\,(\mathbf{p}^2 + \mathbf{M_i^2})\,\mathcal{K}^{-1}(\mathbf{p})$$

M_i: mass of the i-th external particle

$\mathcal{B}$: non-minimal coupling of ϕ_α to the soft graviton

$$\frac{1}{2}\int \frac{\mathbf{d^D q_1}}{(2\pi)^{\mathbf{D}}}\,\frac{\mathbf{d^D q_2}}{(2\pi)^{\mathbf{D}}}\,(2\pi)^{\mathbf{D}}\delta^{(\mathbf{D})}(\mathbf{q_1}+\mathbf{q_2}+\mathbf{k})\,\mathbf{R}_{\mu\rho\nu\sigma}\,\phi_\alpha(\mathbf{q_1})\,\mathcal{B}^{\alpha\beta;\mu\rho\nu\sigma}(\mathbf{q_2})\,\phi_\beta(\mathbf{q_2})$$

$$\mathbf{R}_{\mu\rho\nu\sigma} = (\varepsilon_{\mu\nu}\mathbf{k}_\rho\mathbf{k}_\sigma - \varepsilon_{\mu\sigma}\mathbf{k}_\nu\mathbf{k}_\rho - \varepsilon_{\nu\rho}\mathbf{k}_\sigma\mathbf{k}_\mu + \varepsilon_{\rho\sigma}\mathbf{k}_\mu\mathbf{k}_\nu)$$

– Riemann tensor of the soft graviton

Consistency checks:

1. Known results in string theory and field theory agree with this general formula

2. The final result depends only on the on-shell data of $\Gamma^{\alpha}_{(i)}$ even though the individual terms depend on off-shell data.

Example: Suppose we change $\Gamma^{\alpha}_{(i)} \Rightarrow \Gamma^{\alpha}_{(i)} + (p_i^2 + M_i^2)\, A$

Change in $\partial\Gamma^{\alpha}_{(i)}/\partial p_{i\mu}$ does not vanish on-shell.

However when we add all the terms in the soft graviton theorem, the result is unchanged.