Entanglement in Gauge Theories

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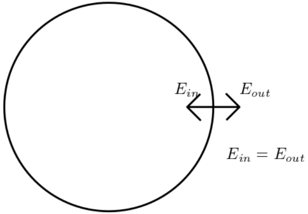
The Problem

Problem: How do you calculate entanglement of a region A with its complement in a gauge theory?

Why it is confusing: Gauge symmetry.
Gauss' law $\nabla \cdot \mathbf{E} = 0$ relates electric field leaving a region to that entering it.
$\Rightarrow$ Degrees of freedom not independent, so Hilbert space doesn't factorise: $\mathcal{H} \neq \mathcal{H}_{in} \otimes \mathcal{H}_{out}$.



Extended Hilbert Space Definiton: Ignore the Problem[2]

(On lattice and in $A_0 = 0$ gauge,
taking region A as a set of links)

Gauge-invariant states satisfy Gauss' law:

$$|\psi\rangle \in \mathcal{H}_{ginv} \quad \Leftrightarrow \quad \nabla \cdot \hat{\mathbf{E}} \, |\psi\rangle = 0. \tag{1}$$

Extended Hilbert Space (EHS) $\mathcal{H}$ has these as well as all non-gauge-invariant states.[1]
EHS doesn't know about Gauss' law $\Rightarrow$ it has tensor factorisation $\Rightarrow$ it has a natural reduced density matrix.

Use this definition for $|\psi\rangle \in \mathcal{H}_{ginv} \subset \mathcal{H}$.

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[1] But no negative-norm states, because of $A_0 = 0$ gauge.

[2] P. V. Buividovich and M. I. Polikarpov 0806.3376;
W. Donnelly 1109.0036; S. Ghosh, RMS and S. P. Trivedi 1501.02593.

Some Properties

1. Natural for emergent gauge theories.[3]
   These are theories that have gauge symmetry only at low energies. (Condensed matter systems.)
2. Given by lattice replica trick[4]
   using Wilson path integral (= no gauge-fixing).
3. Natural continuum limit given by above replica trick,[5]

$$S_{\text{lattice replica}} \xrightarrow{\text{continuum limit}} S_{\text{continuum replica}} + c|G|\frac{A}{\epsilon^{d-2}}. \quad (2)$$

   $|G|$: no. of points in gauge group G ($= \infty$ for cts Lie group).
   Second term is divergent, but proportional to the area so non-universal.
4. Gives replica trick answer $\Rightarrow$ agrees with Ryu-Takayanagi.[6]

[3] L-Y Hung and Y. Wan 1501.04389.
[4] S. Ghosh, RMS and S. P. Trivedi 1501.02593.
[5] RMS and S. P. Trivedi 1608.00353.
[6] A. Lewkowycz and J. M. Maldacena 1304.4926.

So Gauge Symmetry Was a Red Herring?

Gauge symmetry manifests in a decomposition of the entanglement entropy according to the boundary electric flux configuration ("edge modes") $\mathbf{k}$,

$$S_E(A) = -\sum_{\mathbf{k}} p_{\mathbf{k}} \operatorname{Tr} \rho_{\mathbf{k}} \log \rho_{\mathbf{k}} - \sum_{\mathbf{k}} p_{\mathbf{k}} \log p_{\mathbf{k}} - \sum_{\mathbf{k}} p_{\mathbf{k}} \log d_{\mathbf{k}}.^{7} \quad (3)$$

$p_{\mathbf{k}}$: probability of finding electric flux configuration $\mathbf{k}$.
$\rho_{\mathbf{k}}$: density matrix in relevant subsector.
$d_{\mathbf{k}}$: dimension of representation $\mathbf{k}$ (non-Abelian theory).

For $SU(2)$, $\mathbf{k} \sim J^2$ at every boundary point.
The third term comes from the fact that singlet for spin j is like $\sum_m |j, m\rangle \otimes |j, -m\rangle$.

[7] W. Donnelly 1406.7304.

What Does That Mean?

$$S_E(A) = -\sum_{\mathbf{k}} p_{\mathbf{k}} \,\mathrm{Tr}\, \rho_{\mathbf{k}} \log \rho_{\mathbf{k}} - \sum_{\mathbf{k}} p_{\mathbf{k}} \log p_{\mathbf{k}} - \sum_{\mathbf{k}} p_{\mathbf{k}} \log d_{\mathbf{k}}. \quad (4)$$

The three terms are[8]

1. Distillable Entanglement: Actual entanglement $\sim$ no. of Bell pairs equivalent to the state.
2. Classical Correlation: The amount of correlation that arises from equality of electric flux entering and leaving the region.
3. Fusion Term: Comes from correlations of non-gauge-invariant operators. In $SU(2)$, they are the J_zs.

But all three contribute to replica trick answer!

[8] RMS and S. P. Trivedi, 1510.07455
K. van Acoleyen et al, 1511.04369.