Symmetries of Feynman Integrals

Ruth Shir
work with Barak Kol and Philipp Burda

The Hebrew University of Jerusalem

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A Feynman integral

$$I(x) = \int \frac{d^d l_1 \dots d^d l_L}{(k_1^2 - x_1)(k_2^2 - x_2)\dots(k_P^2 - x_P)}$$

$$k_i = A_{ia} l_a + B_{ib} p_b$$

Parameter space

$$X = \{m_1^2, \dots, m_P^2, p_1^2, p_1 \cdot p_2, \dots\} \equiv \{x_1, x_2, \dots\}$$

Symmetries

Linear transformation in the space of loop momenta:

$$l_a = L_{ab} l_b$$

To get useful equations, consider an infinitesimal transformation

$$l'_a = l_a + \epsilon l_b$$

this should not affect the value of the integral

$$I' = I + \delta I = I$$

$$\delta I = \epsilon \int \frac{\partial}{\partial l_a} l_b \tilde{I} = 0$$

We get

- $c(d) I(x)$
- $\int \frac{1}{\ldots (k_i^2 - x_i)^2 \ldots} \to \frac{\partial}{\partial x_i} I(x)$
- $\int \frac{1}{\ldots \cancel{(k_i^2 - x_i)} \ldots (k_j^2 - x_j)^2 \ldots} \to J_{ij}(x)$ degenerate (source) diagrams
- $\int \frac{\text{Num}}{\ldots (k_i - x_i)^2 \ldots}$ we don't want these

The SFI group

Define the space of squares

$$S \equiv \mathrm{Sp}\{k_1^2, \ldots, k_P^2\}$$

We restrict ourselves to the following group of transformations:

SFI symmetry group

$$G \equiv \text{All linear transformations which preserve } S$$

Each generator of $G \to$ PDE in parameter space X

$$c_a I(x) - M_{ai}(x) \frac{\partial}{\partial x_i} I(x) = J_a(x)$$

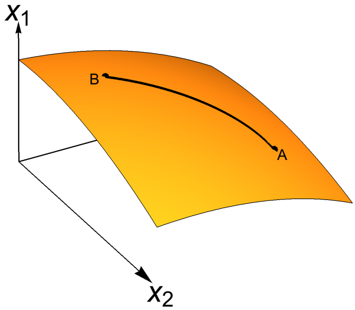
B. Kol 2015, 2016

G orbits

The set of differential operators $D_a(x) = M_{ai}(x)\frac{\partial}{\partial x_i}$ define G orbits in parameter space X.

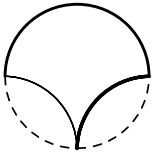
Main idea

- Within a G orbit we can reach any point from any other point via a line integral. B. Kol 2016
- The smaller $\mathrm{codim}(G \text{ orbits})$ the larger the region we could reach in parameter space X.



New Results

Consider the following 3 loop Feynman integral



$$= \int \frac{d^dl_1 d^dl_2 d^dl_3}{(l1^2 - x_1)(l_2^2 - x_2)(l_3^2 - x_3)(l_1 + l_2)^2(l_1 + l_3)^2}$$

Parameter space

$$X = \{x_1, x_2, x_3\}$$

SFI equation system

$$\begin{pmatrix} d-4 \\ d-3 \\ d-3 \end{pmatrix} I(x) - \begin{pmatrix} 2x_1 & x_1 + x_2 & x_1 + x_3 \\ 0 & x_2 - x_1 & 0 \\ 0 & 0 & x_3 - x_1 \end{pmatrix} \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix} I(x) = \begin{pmatrix} J_{12} + J_{13} - J_{42} - J_{53} \\ -J_{12} + J_{42} \\ -J_{13} + J_{53} \end{pmatrix}$$

G orbits

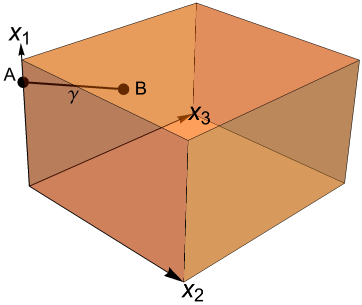
$$\mathrm{codim}(G \text{ orbits}) = 0$$

Homogeneous solution

$$I_0(x) = x_1^{1-d/2}[(x_1 - x_2)(x_1 - x_3)]^{d-3}$$

Full solution $I(x) = I_0(x)c(x)$ from which $\partial_i c(x) = \frac{f_i(J(x))}{I_0(x)}$

$$I(B) = I_0(B)\Big[\int_{\gamma(A\to B)} \frac{f_i(J(x))}{I_0(x)}\, dx_i + \frac{I(A)}{I_0(A)}\Big]$$



P. Burda, B. Kol & RS, 2017