Euler Characteristics of Crepant Resolutions of Weierstrass Models

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Geometric Approach

- By using elliptic fibrations, we geometrically engineer gauge theories in F-theory and M-theory
- M-theory compactified on a Calabi-Yau threefold

  = 5d supersymmetric gauge theory with 8 supercharges

Elliptic fibration	Gauge theory
Codimension 1 singularity	Gauge algebra
Codimension 2 singularity	Representation
Codimension 3 singularity	Yukawa
Crepant resolution	Coulomb Phase
Flop	Phase Transition

Invariance of the Euler characteristic under flops

- Batyrev, Kontsevich – The Euler characteristics of two crepant resolutions of the same variety are the same.

- Compute one crepant resolution explicitly for the Euler characteristic and Hodge numbers for each gauge group from Weierstrass Models.

Algorithm for Euler Characteristics Computations

Step 1. For a Lie group G, determine a singular Weierstrass model with Kodaira fibers associated to G.

Step 2. Determine a crepant resolution of the singular Weierstrass model.

- Compute the total homological Chern class of the crepant resolution.

Step 3. Compute the pushforward formula to push the total Chern class forward to the base.

- The generating function of Euler characteristics is computed.
- For a d-dimensional base, the Euler characteristic is given by the coefficient of t^d in a power series expansion.

Step 4. Compute Euler characteristics for CY 3-folds and 4-folds.

Generating Functions of Euler Characteristics

- Smooth Weierstrass Model: $\chi(Y) = \frac{12L}{1+6L}c(B)$
- Examples (Singular Weierstrass Models):
  - G_2 Weierstrass Model: $\chi(Y) = 12\,\frac{L+2SL-S^2}{(1+S)(1+6L-3S)}c(B)$
  - F_4 Weierstrass Model: $\chi(Y) = 12\,\frac{L+3SL-2S^2}{(1+S)(1+6L-4S)}c(B)$

Hodge Numbers

- Since B is a rational surface, $h^{0,1}(B) = h^{0,2}(B) = 0$.
- Shioda-Tate-Wazir theorem:
  - $h^{1,1}(Y) = h^{1,1}(B) + f + 1.$
  - $h^{2,1}(Y) = h^{1,1}(Y) - \frac{1}{2}\chi(Y).$
  - f is the number of geometrically irreducible fibral divisors not touching the zero section.
  - In particular, with G a simple group, f is the rank of G.

Hodge Numbers and Euler Characteristics of CY 3-folds

Algebra	Group	Kodaira Fiber	$h^{1,1}(Y_3)$	$h^{2,1}(Y_3)$	$\chi(Y_3)$
–	$\{e\}$	I_1	$11 - K^2$	$11 + 29K^2$	$-60K^2$
A_1	$SU(2)$	I_2^s, I_2^{ns}, III, IV^{ns}	$12 - K^2$	$12 + 29K^2 + 15KS + 3S^2$	$-60K^2 - 30KS - 6S^2$
A_2 C_2 G_2	$SU(3)$ $USp(4)$ G_2	I_3^s, IV^s I_4^{ns} I_0^{*ns}	$13 - K^2$	$13 + 29K^2 + 24KS + 6S^2$	$-60K^2 - 48KS - 12S^2$
A_3 B_3	$SU(4)$ $Spin(7)$	I_4^s I_0^{*ss}	$14 - K^2$	$14 + 29K^2 + 32KS + 10S^2$	$-60K^2 - 64KS - 20S^2$
D_4 F_4	$Spin(8)$ F_4	I_0^{*s} IV^{*ns}	$15 - K^2$	$15 + 29K^2 + 36KS + 12S^2$	$-60K^2 - 72KS - 24S^2$
A_4	$SU(5)$	I_5^s	$15 - K^2$	$15 + 29K^2 + 40KS + 15S^2$	$-60K^2 - 80KS - 30S^2$
D_5	$Spin(10)$	I_1^{*s}	$16 - K^2$	$16 + 29K^2 + 42KS + 16S^2$	$-60K^2 - 84KS - 32S^2$
E_6	E_6	IV^{*s}	$17 - K^2$	$17 + 29K^2 + 45KS + 18S^2$	$-60K^2 - 90KS - 36S^2$
E_7	E_7	III^*	$18 - K^2$	$18 + 29K^2 + 49KS + 21S^2$	$-60K^2 - 98KS - 42S^2$
E_8	E_8	II^*	$19 - K^2$	$19 + 29K^2 + 60KS + 30S^2$	$-60K^2 - 120KS - 60S^2$
A_1	$SO(3)$	I_2^{ns}	$12 - K^2$	$12 + 17K^2$	$-36K^2$
B_2	$SO(5)$	I_4^{ns}	$14 - K^2$	$14 + 9K^2$	$-20K^2$
A_3	$SO(6)$	I_4^s	$14 - K^2$	$14 + 5K^2$	$-12K^2$

Euler Characteristics of CY 4-folds

Models	$\chi(Y_4)$, Euler characteristic
Smooth Weierstrass	$12c_1c_2+360c_1^3$
SU(2)	$6(2c_1c_2+60c_1^3-49c_1^2S+14c_1S^2-S^3)$
SU(3) or USp(4) or G_2	$12(c_1c_2+30c_1^3-38c_1^2S+16c_1S^2-2S^3)$
SU(4) or Spin(7)	$12(3c_1c_2+30c_1^3-50c_1^2S+28c_1S^2-5S^3))$
Spin(8) or F_4	$12(c_1c_2+30c_1^3-54c_1^2S+32c_1S^2-6S^3)$
SU(5)	$3(4c_1c_2+120c_1^3-250c_1^2S+175c_1S^2-40S^3)$
Spin(10)	$12(c_1c_2+30c_1^3-63c_1^2S+44c_1S^2-10S^3)$
E_6	$3(4c_1c_2+120c_1^3-258c_1^2S+183c_1S^2-42S^3)$
E_7	$6(2c_1c_2+60c_1^3-135c_1^2S+100c_1S^2-24S^3)$
E_8	$12(c_1c_2+30c_1^3-80c_1^2S+70c_1S^2-20S^3)$
SO(3)	$12c_1(c_3-48c_1^3+-4c_1c_2)$
SO(5)	$4c_1(3c_3-28c_1^3-8c_1c_2)$
SO(6)	$12c_1(-4c_1^3-2c_1c_2+c_3)$

Note: Matches with Blumenhagen-Grimm-Jurke-Weigand (BGJW) conjecture.

Intrilligator-Morrison-Seiberg Prepotential for a 5d Gauge Theory

- The Intrilligator-Morrison-Seiberg prepotential for a 5d gauge theory:

$$6\mathscr{F}_{IMS} = \frac{1}{2}\left(\sum_{\alpha} |\langle\alpha,\phi\rangle|^3 - \sum_{\mathbf{R}_i} \sum_{\varpi\in W_i} n_{\mathbf{R}_i} |\langle\varpi,\phi\rangle|^3\right)..$$

- Consider an exceptional gauge group F_4 :

$$\begin{aligned}
6\mathscr{F}_{IMS} = &-8\left(n_{52}-1\right)\phi_1^3 - 8\left(n_{52}-1\right)\phi_2^3 - 8\left(n_{52}+n_{26}-1\right)\phi_3^3 - 8\left(n_{52}+n_{26}-1\right)\phi_4^3 \\
&- 3\left(-n_{52}+n_{26}+1\right)\phi_1^2\phi_2 + 3\left(n_{52}+n_{26}-1\right)\phi_1\phi_2^2 \\
&+ 12\left(-n_{52}+n_{26}+1\right)\phi_2\phi_3^2 - 6\left(-3n_{52}+n_{26}+3\right)\phi_2^2\phi_3 \\
&+ 6\left(-3n_{52}+n_{26}+3\right)\phi_3\phi_4^2 + 24\left(n_{52}-1\right)\phi_3^2\phi_4 .
\end{aligned}$$

Triple Intersection Numbers

- The triple intersection numbers of the elliptic fibration defined by the crepant resolution of the F_4-model:

$$\varphi_*\Big((\sum D_a \phi_a)^3 \cdot \varphi^* M \Big) = -8(g-1)\phi_0^3 - 3(-4g+4+S^2)\phi_0^2\phi_1 + 3(-2g+2+S^2)\phi_1^2\phi_0$$
$$-8(g-1)\phi_1^3 - 8(g-1)\phi_2^3 + 8(4g-4-S^2)\phi_3^3 + 8(4g-4-S^2)\phi_4^3$$
$$3(6g-6-S^2)\phi_1^2\phi_2 + 3(-4g+4+S^2)\phi_1\phi_2^2$$
$$+6(8g-8-S^2)\phi_2^2\phi_3 + 12(-6g+6+S^2)\phi_2\phi_3^2 + 24(g-1)\phi_3^2\phi_4 + 6(-8g+8+S^2)\phi_3\phi_4^2.$$

- It matches the Intrilligator-Morrison-Seiberg prepotential when

$$n_{\mathbf{52}} = g, \qquad n_{\mathbf{26}} = 5(1-g) + S^2.$$

adjoint *fundamental*

- When $g=0$ and $S^2=-5$, we have no representations even though codim 2 singularities exist. In such a case, we say that the representation is *frozen*.

Conclusion

- We found an easy systematic algorithm to compute the Euler characteristics for elliptic fibrations of arbitrary dimensions that are not necessarily CY.
- M-theory compactified on a CY 3-fold gives a 5d supersymmetric gauge theory with a gauge group G and its representation R. Euler characteristics and Hodge numbers are computed for these CY 3-folds for these gauge theories.
- M-theory compactified on a CY 4-fold gives a 3d gauge theory. The Euler characteristics for the CY 4-folds are also computed.
- Intrilligator-Morrison-Seiberg prepotential is computed to describe the Coulomb branch of the 5d gauge theory. We compute explicitly the number of representations by matching the prepotential with the triple intersection numbers computed geometrically. The numbers we find match those obtained in a 6d theory using anomaly cancellations.

Thank you for listening!