Time-reversal anomaly of 2+1d topological phases

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in collaboration with
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[1610.07010, 1611.01601, ...]

Strings 2017, Tel Aviv

The first quantum anomaly we learn is about a

- continuous symmetry,
- in even dimensions,
- with massless excitations,
- which are fermions.

Today I will talk about quantum anomaly of a

- continuous symmetry,
- in even dimensions,
- with massless excitations,
- which are fermions.

Today I will talk about quantum anomaly of a

- **discrete** symmetry,
- in even dimensions,
- with massless excitations,
- which are fermions.

Today I will talk about quantum anomaly of a

- **discrete** symmetry,
- in **odd** dimensions,
- with massless excitations,
- which are fermions.

Today I will talk about quantum anomaly of a

- **discrete** symmetry,
- in **odd** dimensions,
- **without any** massless excitations,
- which are fermions.

Today I will talk about quantum anomaly of a

- **discrete** symmetry,
- in **odd** dimensions,
- **without any** massless excitations,
- **and with anyons**.

For example, the 3d Chern-Simons theory

$$\mathbf{U}(\boldsymbol{n})_{\boldsymbol{2n}}$$

has a **secret parity symmetry**, but **with an anomaly**

$$\boldsymbol{\nu = \pm 2 \in \mathbb{Z}_{16}}.$$

I'd like to explain this to you, using the rest of my time today.

Note 1: I won't distinguish parity and time-reversal unless necessary, which is OK thanks to CPT.

Note 2: I won't be careful about the **almost trivial spin TQFT** part in the talk, if you know what I mean.

Note 3: I will concentrate on one particular example for illustration, but the formalism is general.

Why should you care?

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I really don’t know.

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Isn't it pretty funny that $\mathrm{U}(n)_{2n}$ is parity symmetric?

Why should you care?

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Isn't it pretty funny that $\mathrm{U}(n)_{2n}$ is parity symmetric?

That was a sufficient motivation for me to study it.

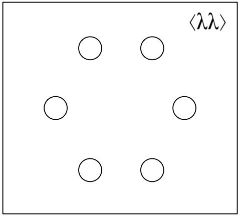
So, let's dive into it.

Why is the 3d $\mathrm{U}(n)_{2n}$ Chern-Simons theory parity symmetric?

There're many ways to show this but
let me use a method which appeals to a 4d SUSY person like me...

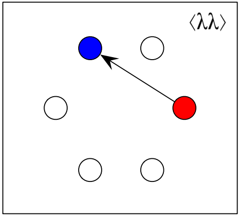
Consider 4d $\mathcal{N}=1$ $\mathbf{SU}(X)$ super Yang-Mills.

It has X vacua.



Here $X = 6$.

There are domain walls connecting different vacua of 4d $\mathbf{SU}(X)$ theory.



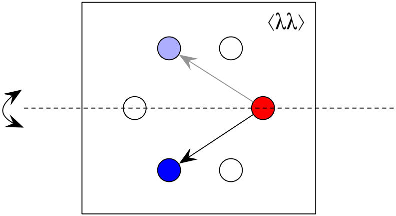
When n steps apart, the worldvolume theory is 3d $\mathcal{N}=1$ supersymmetric

$$\mathbf{U}(n)_X$$

Chern-Simons theory (+ the center-of-mass mode.)
[Acharya-Vafa hep-th/0103011]

In the example above we have $\mathbf{U}(2)_6$.

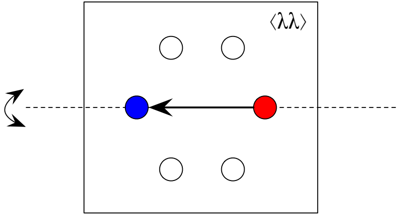
Spacetime parity sends $\langle\boldsymbol{\lambda\lambda}\rangle$ to $\overline{\langle\boldsymbol{\lambda\lambda}\rangle}$.



It exchanges

$$\mathrm{U}(n)_X \leftrightarrow \mathrm{U}(X-n)_X.$$

Therefore, when $X = 2n$, we have:



meaning that

$$U(n)_{2n} \leftrightarrow U(n)_{2n}$$

should be parity symmetric.

The parity transformation

$$\mathrm{U}(n)_X \leftrightarrow \mathrm{U}(X-n)_X.$$

is in fact the **level-rank duality**. [Hsin-Seiberg, 1607.07457]

So far I've been using the $\mathcal{N}=1$ convention for levels.

In the TQFT convention, we have

$$\mathrm{U}(n)_X = \frac{\mathrm{SU}(n)_{X-n} \times \mathrm{U}(1)_{nX}}{\mathbb{Z}_n}$$

and

$$\mathrm{U}(X-n)_X = \frac{\mathrm{SU}(X-n)_n \times \mathrm{U}(1)_{(X-n)X}}{\mathbb{Z}_{X-n}}$$

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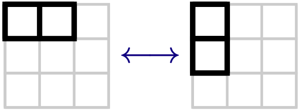
$$\mathrm{U}(X-n)_X = \frac{\mathrm{SU}(X-n)_n \times \mathrm{U}(1)_{(X-n)X}}{\mathbb{Z}_{X-n}}$$

Let's concretely check that $\mathrm{U}(n)_{2n}$ is parity symmetric.

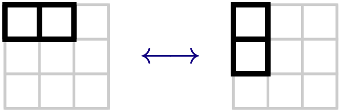
The anyons/quasiparticles/lines of $\mathrm{U}(n)_X$ are specified by Young diagrams which fit in a box of size $n \times (X - n)$.

The level-rank duality is the transpose.

Let's take $n = 3$, $X = 2n = 6$. An example:



Computing anyons' spins $\boldsymbol{h}$ using standard formulas, we find

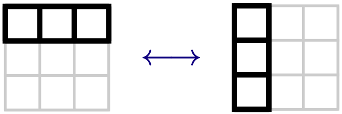


$$\boldsymbol{h = 2/3} \quad \longleftrightarrow \quad \boldsymbol{h = 1/3}$$

This is as it should be, since the parity should do

$$\boldsymbol{h} \longleftrightarrow \boldsymbol{-h}.$$

Let's do another example.



$$h = 1/4 \longleftrightarrow h = 1/4$$

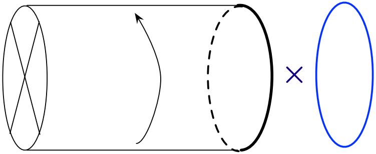
This is consistent with

$$h \longleftrightarrow -h$$

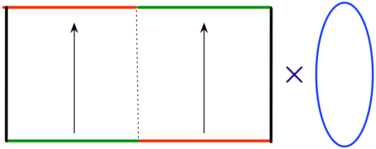
because this is a **spin TQFT**, for which
h of anyons in the NS sector is defined only mod $1/2$.

Physically, there are very heavy but dynamical fermions in the system, which can change the spin of a quasiparticle by $1/2$.

Let's have some fun by putting $\mathbf{U}(n)_{2n}$ on non-orientable spacetimes.
Consider the Möbius strip times a circle.



or equivalently

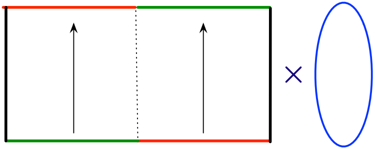


This has a torus boundary, and therefore creates a state in $\mathcal{H}_{T^2}$.

Call it a crosscap state $|\text{crosscap}\rangle$.

To determine the state

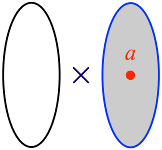
$$|\text{crosscap}\rangle =$$



,

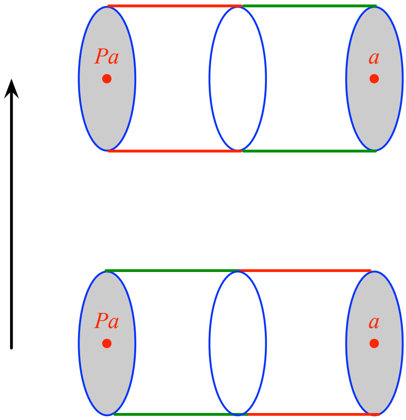
let us glue it to

$$|\boldsymbol{a}\rangle =$$



.

After some mental gymnastics, the geometry is



meaning that

$$\langle a|\text{crosscap}\rangle = \mathrm{tr}\, P \text{ on Hilb. on } S^2 \text{ with } a \text{ and } Pa.$$

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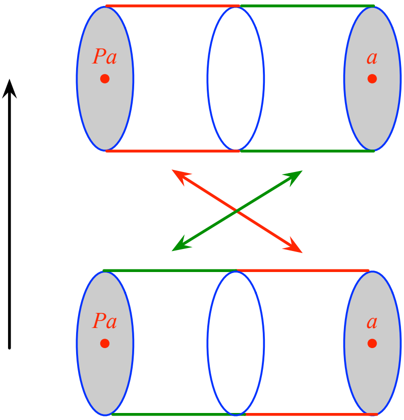

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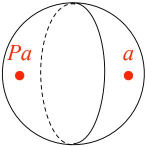
So we have

$$\langle a|\text{crosscap}\rangle = \mathrm{tr}\, P \text{ on Hilb. on } S^2 \text{ with } a \text{ and } Pa$$
$$= \begin{cases} \pm 1 & \text{if } a = \overline{Pa}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\overline{Pa} = Ta$ due to the CPT theorem. So

$$|\text{crosscap}\rangle = \sum_{a=Ta} \pm |a\rangle$$

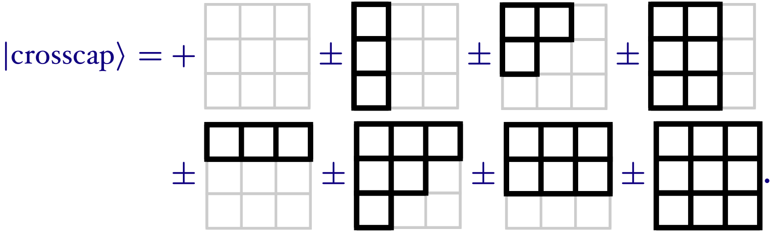
where $\pm$ in front of $|a\rangle$ specifies the P eigenvalue of the state



on S^2.

[Barkeshli-Bonderson-Cheng-Jian-Walker, 1612.07792]

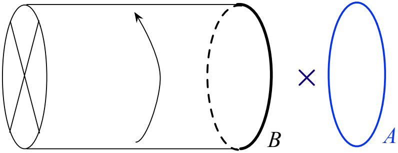
For $\mathbf{U(3)_6}$ we have



How do we determine the signs?

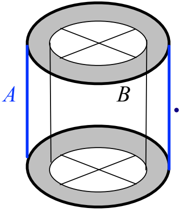
So far we displayed the crosscap state as

$$|\text{crosscap}\rangle =$$



but let's now view it as

$$|\text{crosscap}\rangle =$$



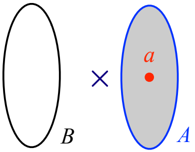
The geometry doesn't change under

$$\boldsymbol{B} \mapsto \boldsymbol{B}, \qquad \boldsymbol{A} \mapsto \boldsymbol{A} + \boldsymbol{B}.$$

The action of

$$B \mapsto B, \qquad A \mapsto A + B$$

in the basis we're using



is $S^{-1}TS$.

This means

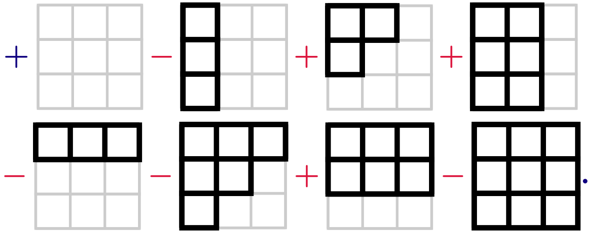
$$S^{-1}TS\,|\text{crosscap}\rangle \propto |\text{crosscap}\rangle$$

where

$$|\text{crosscap}\rangle = \sum_{a=Ta} \pm\,|a\rangle\,.$$

For $\mathbf{U(3)_6}$ this is enough to fix the signs essentially uniquely:

$|\text{crosscap}\rangle =$



with

$$S^{-1}TS\,|\text{crosscap}\rangle = \exp\left(\frac{2\pi i \cdot 2}{16}\right)|\text{crosscap}\rangle\,.$$

This phase is a manifestation of the anomaly of spatial parity $\sim$ time-reversal.

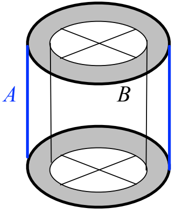
The anomalous phase

$$S^{-1}TS\,|\text{crosscap}\rangle = \exp\left(\frac{2\pi i \cdot 2}{16}\right)|\text{crosscap}\rangle\,.$$

is associated to the operation

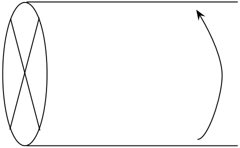
$$B \mapsto B, \qquad A \mapsto A + B$$

in the geometry



.

Regarding $\boldsymbol{A}$ as the time direction, we see the system on

$\mathbb{R}_{\boldsymbol{t}} \times$ 

has the momentum

$$p = \frac{\mathbf{2}}{\mathbf{16}} \mod 1$$

which is the conserved quantity associated to the isometry.

In a non-anomalous theory, we have

$$p = n \in \mathbb{Z}.$$

This is because the 2π rotation should not do anything:

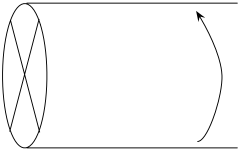
$$\exp\left[2\pi i p\right] = 1.$$

In an anomalous theory, this might not hold, because of phase ambiguity:

$$\exp\left[2\pi i p\right] \neq 1.$$

[Cho-Hsieh-Morimoto-Ryu, 1501.07285]

For example, on

$\mathbb{R}_t \times$  ,

a massless Majorana fermion in 3d has [Hsieh-Cho-Ryu,1503.01411]

$$p = \frac{1}{16} \mod 1$$

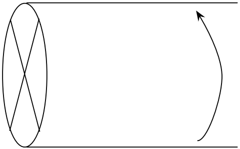
and we found that $\mathbf{U(3)_6}$ has twice the anomaly

$$p = \frac{2}{16} \mod 1.$$

ν Majorana fermions in 3d have the anomalous momentum

$$p = \frac{\nu}{16} \mod 1$$

on

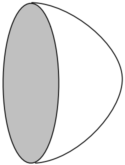
$\mathbb{R}_t \times$  .

So $\nu = 16$ fermions do not manifest anomalies in this geometry.

In fact this is a general feature:

Parity anomaly of this type of systems is a $\mathbb{Z}_{16}$-valued quantity.

To explain this, let us pause for a moment and consider $\mathbf{U(1)}$ anomaly in 4d, which is characterized by the anomaly inflow

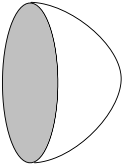
$\boldsymbol{k}$ 4d chiral fermions $\rightarrow$  $\leftarrow \exp[2\pi i k \int_X A \wedge F \wedge F]$

The anomaly is characterized by $\boldsymbol{k} \in \mathbb{Z}$ since

$$\exp(2\pi i \int_X A \wedge F \wedge F)$$

is a general complex number of absolute value 1.

In our case, the anomaly is canceled by the anomaly inflow

k 3d Majorana fermions $\rightarrow$  $\leftarrow \exp[2\pi i \nu \eta_X]$

where the 4d bulk term is the Atiyah-Patodi-Singer η invariant.
[Kapustin-Thorngren-Turzillo-Wang, 1406.7329] [Witten, 1605.02391]

The anomaly is characterized by $\nu \in \mathbb{Z}_{16}$, since

$$\exp(2\pi i \eta_X)$$

of any closed manifold is a 16-th root of unity.

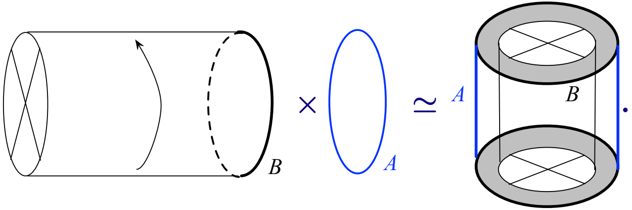
Summarizing, we noted that the 3d Chern-Simons theory

$$\mathbf{U}(n)_{2n}$$

is secretly parity invariant, but has an anomaly

$$\nu = 2 \in \mathbb{Z}_{16}.$$

We arrived at this result by considering a state



Where to go from here?

Firstly, there are things to be cleaned up:

- 3d TQFT on oriented manifold : [Moore-Seiberg "RCFT"] and [Witten, "QFT and the Jones polynomial"].
- 3d TQFT on oriented manifold **with spin structure** : [Bruillard-Galindo-Hagge-Ng-Plavnik-Rowell-Wang, 1603.09294], [Bhardwaj-Gaiotto-Kapustin, 1605.01640].
- 3d TQFT on **non-orientable** manifold : [Barkeshli-Bonderson-Cheng-Jian-Walker, 1612.07792]

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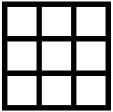
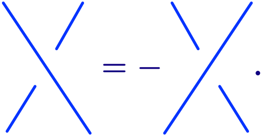
- 3d TQFT on **non-orientable** manifold : [Barkeshli-Bonderson-Cheng-Jian-Walker, 1612.07792]

But we do not yet have a definitive treatment of 3d TQFT on **non-orientable** manifold **with pin structure**.

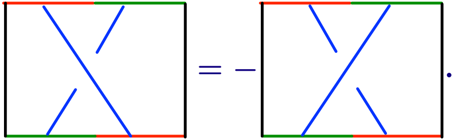
Somebody has to do that.

It's kind of tricky. For example, in **$U(3)_6$**, the anyons of type 

have the braiding

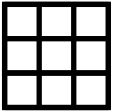


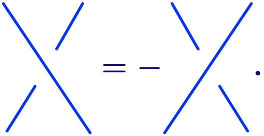
But if we put this on the crosscap it is problematic.



This only happens with **non-orientable** + **spin**.

And this makes my head hurt.

It's kind of tricky. For example, in $\mathbf{U(3)_6}$, the anyons of type  have the braiding



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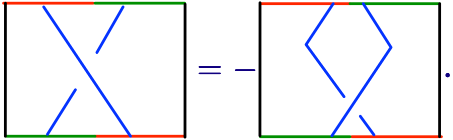

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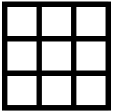
It's kind of tricky. For example, in $\mathbf{U(3)_6}$, the anyons of type 

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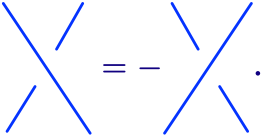

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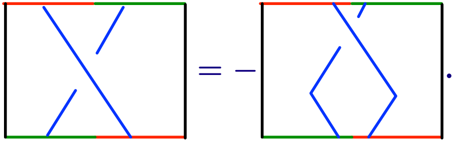

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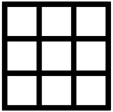
It's kind of tricky. For example, in $\mathbf{U(3)_6}$, the anyons of type 

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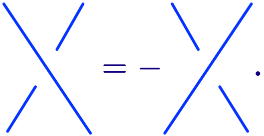

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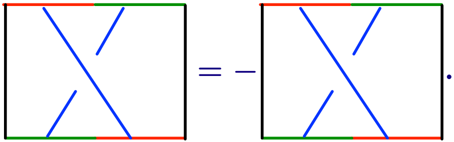

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Secondly, more importantly, suppose you want to use matching of these subtler/new anomalies to constrain the dynamics.
cf. [Gaiotto-Kapustin-Komargodski-Seiberg, 1703.00501]

If the anomaly can be realized by a TQFT,
you can just add it to match the missing anomaly.

So you need to know when an anomaly can be realized by a TQFT.

That's it! Thanks for your attention.