New $\mathcal{N} = 4$ theories in four dimensions

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Classifying $\mathcal{N} = 4$ theories

Known $\mathcal{N} = 4$ theories in four dimensions are classified by a choice of gauge group G (with algebra $\mathfrak{g}$), and some discrete θ angles. [Aharony, Seiberg, Tachikawa '13]

A prototypical example is $\mathfrak{su}(2) \to \{SU(2), SO(3)_\pm = (SU(2)/\mathbb{Z}_2)_\pm\}$ (triplet of $SL(2,\mathbb{Z})$).

In the case of $\mathcal{N} = 4$ the structure of point operators is insensitive to the choice of theory, they only depend on $\mathfrak{g}$. One can detect the difference by studying the partition function on four-manifolds with $H^2(\mathcal{M}_4, C) \neq 0$, or by studying the properties and correlators of extended operators.

Holography and relative theories

What is the holographic interpretation of the possible variants?

We view the possible theories on the boundary as states in the Hilbert space of the bulk theory, taking the radial direction as "time". [Friedan, Shenker '87], [Verlinde '88], [Moore, Seiberg '88], [Witten '89], ..., [Witten '98], ..., [Belov, Moore], ...

The partition function on each theory/state $|\psi\rangle$ is

$$Z_\psi(\tau) = \langle\psi|Z(\tau)\rangle \tag{1}$$

for some $|Z(\tau)\rangle$ to be described momentarily, the "partition vector".

Theories that can be understood in such a way are known as *relative QFTs*. [Freed, Teleman '12] Famous examples are the chiral boson in 2d, the $(0,2)$ theory in 6d, and IIB string theory.

4d $\mathcal{N}=4$ SYM can also be understood in this way.

Quantization of the bulk TQFT

(Following [Witten '98])

The reduction of IIB on S^5 gives an effective action

$$L_{\mathsf{CS}} = \frac{N}{2\pi i} \int_{X_5} B_2 \wedge dC_2\,. \tag{2}$$

In order to specify the boundary conditions, in addition to specifying the vevs of local gauge invariant operators, we need to specify

$$\alpha = \int_S B_2 \quad ; \quad \beta = \int_S C_2 \tag{3}$$

for any $S \subset \mathcal{M}_4$ near the boundary, $X_5 \approx \mathbb{R} \times \mathcal{M}_4$. The equations of motion are

$$dB_2 = dC_2 = 0 \tag{4}$$

and B_2, C_2 are canonically conjugate ($B = C = 0$ is disallowed!):

$$[B_{ij}(x), C_{kl}(y)] = -\frac{2\pi i}{N} \epsilon_{ijkl} \delta^4(x-y)\,. \tag{5}$$

Quantization of the bulk TQFT

(Following [Witten '98])

Define operators measuring the flux

$$\Phi_{\mathsf{RR}}(S) = \exp\left(i\int_S C_2\right) \quad ; \quad \Phi_{\mathsf{NS}}(T) = \exp\left(i\int_T B_2\right) . \quad (6)$$

They do not commute:

$$\Phi_{\mathsf{RR}}(S)\Phi_{\mathsf{NS}}(T) = \Phi_{\mathsf{NS}}(T)\Phi_{\mathsf{RR}}(S)\exp\left(\frac{2\pi i}{N} S\cdot T\right) . \quad (7)$$

The inequivalent operators are parameterized by classes in $H_2(\mathcal{M}_4, \mathbb{Z}_N)$, so the group of operators acting on the Hilbert space is the finite Heisenberg group W in

$$0 \to \mathbb{Z}_N \to W \to H^2(\mathcal{M}_4, \mathbb{Z}_N)_{\mathsf{NS}} \times H^2(\mathcal{M}_4, \mathbb{Z}_N)_{\mathsf{RR}} \to 0 . \quad (8)$$

Quantization of the bulk TQFT

(Following [Witten '98])

Up to redefinitions W has a single representation. It can be constructed starting from a maximal isotropic subspace $\mathcal{I}$, i.e. a maximal commuting set of operators $\Phi(w)$.

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For any such $\mathcal{I}$ there is a unique state $|\Omega_{\mathcal{I}}\rangle$ (up to normalization) such that

$$\Phi(w)\,|\Omega_{\mathcal{I}}\rangle = |\Omega_{\mathcal{I}}\rangle \quad \forall w \in \mathcal{I}\,. \tag{9}$$

Quantization of the bulk TQFT

(Following [Witten '98])

There is also a basis $\{|w\rangle_{\mathcal{I}}\}$ that diagonalizes $\Phi(w)$ for $w \in \mathcal{I}$:

$$\Phi(w)\left|w'\right\rangle_{\mathcal{I}} = \omega_N^{w\cdot w'}\left|w'\right\rangle_{\mathcal{I}} \qquad \text{for } w \in \mathcal{I} \qquad (10)$$

$$\Phi(w)\left|w'\right\rangle_{\mathcal{I}} = \left|w' + w\right\rangle_{\mathcal{I}} \qquad \text{for } w \in \mathcal{J} \qquad (11)$$

where $\omega_N = \exp(2\pi i/N)$, and $\mathfrak{H} = \mathcal{I} \oplus \mathcal{J}$.

We take the choice of duality frame where

$$|Z(\tau)\rangle = \sum_{w \in H^2(\mathcal{M}_4, \mathbb{Z}_N)} Z_w(\tau)\,|w\rangle_{\mathsf{NS}} \qquad (12)$$

with $Z_w(\tau)$ the partition function of $\mathcal{N} = 4\ SU(N)/\mathbb{Z}_N$ in the sector with Stiefel-Whitney class w. [Witten '09] [Tachikawa '13]

So the choice of $\mathcal{I}$ specifies over which $w \in H^2(\mathcal{M}_4, \mathbb{Z}_N)$ to sum.

The $(0,2)$ viewpoint

It is very natural to rephrase the previous discussion in terms of the 6d $(0,2)$ A_{N-1} theory. [Witten '98] Holographically, the key term is

$$\mathcal{L} = N \int_{X_7} C_3 \wedge dC_3 + \dots \tag{13}$$

In this context the choice of theory is given by a maximal commuting sublattice of $H^3(\mathcal{M}_6, \mathbb{Z}_N)$.

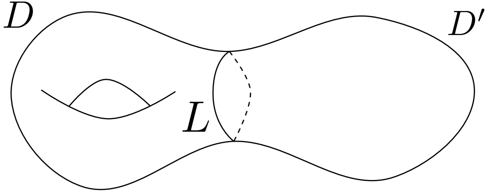
All the previous discussion can be reproduced in this language, if we choose $\mathcal{M}_6 = \mathcal{M}_4 \times T^2$. The theories in [Aharony, Seiberg, Tachikawa '13] appear as polarizations of the T^2 factor. [Tachikawa '13]

Spectrum of extended operators

A basic distinction for line operators is whether a surface attached to the line needs to be specified for defining the line operator. If not, we say that the operator is *genuine*. [Kapustin,Seiberg '14]

If we have a gauge group G, Wilson lines in representations of G are genuine, and we then find the genuine magnetically charged lines by imposing mutual locality.

This can also be understood without reference to a gauge group: consider two surface operators D, D' with boundary L. We say that L is genuine if $S = D - D'$ cannot measure the fluxes in $\mathcal{I}$, for any choice of D, D'.



A self-dual $\mathcal{N} = 4$ $\mathfrak{su}(2)$ theory?

Recently Argyres and Martone proposed the existence of a $\mathcal{N} = 4$ theory with algebra $\mathfrak{su}(2)$ which is completely invariant under $SL(2, \mathbb{Z})$, as part of their work on the classification of rank-1 $\mathcal{N} = 2$ SCFTs. [Argyres, Martone '16]

If it exists it implies the existence of new $\mathcal{N} = 2$ and $\mathcal{N} = 3$ theories via discrete gaugings of various subgroups of $SL(2, \mathbb{Z})$.

Such a theory should be associated with a polarization $\mathcal{I} = \mathcal{I}^{(2)}_{\text{NS}} \oplus \mathcal{I}^{(2)}_{\text{RR}}$ of $H^2(\mathcal{M}_4, \mathbb{Z}_N)_{\text{NS}} \times H^2(\mathcal{M}_4, \mathbb{Z}_N)_{\text{RR}}$, with $\mathcal{I}^{(2)}$ a maximal isotropic sublattice of $H^2(\mathcal{M}_4, \mathbb{Z}_2)$ of half-rank.

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$\mathcal{I}^{(2)}$ exists for any closed smooth orientable Spin fourfold (without torsion).

On the intersection form of four-manifolds

We take $\mathcal{M}_4$ to be closed, orientable, smooth and Spin (and having no torsion in $H^3(\mathcal{M}_4, \mathbb{Z})$, to make our lifes simpler).

A result of Donaldson [Donaldson '83] [Donaldson '87] shows that any such manifold has intersection form over $\mathbb{Z}$ given in some basis by

$$Q = (-\mathcal{C}(E_8))^{\oplus m} \oplus H^{\oplus n} \tag{14}$$

with $\mathcal{C}(E_8)$ the Cartan matrix of E_8 and

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{15}$$

So the problem of existence of a $\mathcal{I}^{(2)}$ reduces to the existence of a half-rank polarization of $\mathcal{C}(E_8)$...

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So the problem of existence of a $\mathcal{I}^{(2)}$ reduces to the existence of a half-rank polarization of $\mathcal{C}(E_8)$... Which obviously does not exist (in general) over $\mathbb{Z}$, since $\mathcal{C}(E_8)$ is definite positive!

Existence of $\mathcal{I}^{(2)}$

No obvious no-go in $\mathbb{Z}_2$: signature and positivity are ill-defined.

$$C(E_8) = \begin{pmatrix} 2 & 0 & \mathbf{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & \mathbf{-1} & 0 & 0 & 0 & 0 \\ \mathbf{-1} & 0 & 2 & \mathbf{-1} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{-1} & \mathbf{-1} & 2 & \mathbf{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{-1} & 2 & \mathbf{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{-1} & 2 & \mathbf{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{-1} & 2 & \mathbf{-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{-1} & 2 \end{pmatrix} . \qquad (16)$$

Existence of $\mathcal{I}^{(2)}$

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$$C(E_8)/2\mathbb{Z} = \begin{pmatrix} 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \end{pmatrix} . \qquad (17)$$

Existence of $\mathcal{I}^{(2)}$

An easy exercise shows that $S(C(E_8)/2\mathbb{Z})S^t = H^{\oplus 4}$, with

$$S = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{18}$$

So, over $\mathbb{Z}_2$

$$Q = (-\mathcal{C}(E_8))^{\oplus m} \oplus H^{\oplus n} \to H^{\oplus(4m+n)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\oplus(4m+n)} \tag{19}$$

as already remarked by [Vafa, Witten '94].

Partition function for $\mathcal{I} = \mathcal{I}^{(2)}_{\textbf{NS}} \oplus \mathcal{I}^{(2)}_{\textbf{RR}}$

$$|\Omega\rangle = \sum_{w \in H^2(\mathcal{M},\mathbb{Z}_2)} c_w \, |w\rangle \tag{20}$$

in the Stiefel-Whitney (NS) basis. For $u \in \mathcal{I}^{(2)}_{\mathsf{NS}}$ we have

$$\Phi(u)\,|\Omega\rangle = \sum_{w \in H^2(\mathcal{M},\mathbb{Z}_2)} (-1)^{u \cdot w} c_w \, |w\rangle \stackrel{!}{=} |\Omega\rangle \tag{21}$$

so $c_w = 0$ if $w \notin \mathcal{I}^{(2)}$. Then, for $u \in \mathcal{I}^{(2)}_{\mathsf{RR}}$

$$\Phi(u)\,|\Omega\rangle = \sum_{w \in \mathcal{I}^{(2)}} c_w \, |w + u\rangle \stackrel{!}{=} |\Omega\rangle \tag{22}$$

implies that $c_v = c_w$ for all $v, w \in \mathcal{I}^{(2)}$. So we have

$$\langle Z(\tau)|\Omega\rangle = \sum_{w \in \mathcal{I}^{(2)}} Z_w(\tau)\,. \tag{23}$$

Partition function on $K3$

As a quick check, we can compute explicitly the partition function on $K3$. [Vafa, Witten '94] We have that

$$Z_w(\tau) = \begin{cases} Z_0 = \frac{1}{4}G(q^2) + \frac{1}{2}\left[G(q^{1/2}) + G(-q^{1/2})\right] & \text{if } w = 0 \\ Z_e = \frac{1}{2}\left[G(q^{1/2}) + G(-q^{-1/2})\right] & \text{if } w \neq 0, \mathcal{P}(w) = 0 \\ Z_o = \frac{1}{2}\left[G(q^{1/2}) - G(-q^{-1/2})\right] & \text{if } w \neq 0, \mathcal{P}(w) = 2 \end{cases} \tag{24}$$

with $G(q) \equiv \eta(q)^{-24}$, $q = \exp(2\pi i \tau)$, and $\mathcal{P}(w)$ the Pontryagin square.

We find that if $\mathcal{P}(w) = 0$ for all $w \in \mathcal{I}^{(2)}$ (✓), the partition function

$$Z_{\mathcal{I}^{(2)}}(\tau) = Z_0(\tau) + (2^{11} - 1)Z_e(\tau) \tag{25}$$

is $SL(2, \mathbb{Z})$ invariant. (Or rather, appropriately covariant.)

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In fact, this is the only combination of the form

$$Z_{(a,b,c)}(\tau) = aZ_0(\tau) + bZ_e(\tau) + cZ_o(\tau) \tag{26}$$

which is $SL(2,\mathbb{Z})$ invariant, up to overall normalization.

Extension to other primes

The construction works for any prime N. Consider

$$G[N] = \begin{pmatrix} \mathbb{I}_4 + N(2N+1)\mathbb{J}_4 & 2N^2\mathbb{J}_4 \\ 2N^2\mathbb{J}_4 & \mathbb{I}_4 + N(2N-1)\mathbb{J}_4 \end{pmatrix} . \qquad (27)$$

with $\mathbb{I}_4$ the 4×4 identity matrix, and $(\mathbb{J})_{ij} = 1$. It is easy to check that $\det(G[N]) = 1$, the matrix is positive definite, and that for $N \in 2\mathbb{Z} + 1$ the associated bilinear form is even. So under a change of basis in $\mathbb{Z}$ this must be equivalent to $\mathcal{C}(E_8)$. On the other hand,

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Since, modulo a prime $a^2 + b^2 = -1$ always has a solution, we can introduce $\widetilde{e}_1 = ae_1 + be_2$, $\widetilde{e}_2 = -be_1 + ae_2$ to obtain

$$G[N] = \begin{pmatrix} \mathbb{I}_4 & \\ & -\mathbb{I}_4 \end{pmatrix} \qquad (29)$$

Proving $G[N] = H^{\oplus 4}$ mod N from here is trivial.

Conclusions

- I showed the existence of new polarizations (giving self-dual theories) on smooth oriented closed Spin manifolds without torsion. The theories are best understood in the context of relative QFTs, or holography.
- So "non-genuine", but still interesting, and useful:
  - Starting point for constructing new $\mathcal{N}=2$ and $\mathcal{N}=3$ theories. [Argyres, Martone '16]
  - Duality defects from the A_N $(0,2)$ theory on $T^2 \to X_6 \to \mathcal{B}_4$, with II, IV, II^* or IV^* singularities over divisors of $\mathcal{B}_4$.
  - Modular invariant theories on T^2 from reduction of the A_N $(0,2)$ theory on the $\mathcal{M}_4$ factor of $\mathcal{M}_4 \times T^2$.

Open questions

- Global p-form symmetries.
- Existence of the isotropic polarization of $C(E_8)$ for all N.
- Classification of variants, other than the self-dual case.
- Incorporate torsion in $H^3(\mathcal{M}_4, \mathbb{Z})$.
- Other gauge algebras.
- Gluing.

תודה רבה

Reproducing the AST classification

The classification of [Aharony, Seiberg, Tachikawa '13] can be understood from this viewpoint [Tachikawa '13]:

Consider the polarization $\mathcal{I}_{T^2} \otimes H^2(\mathcal{M}_4, \mathbb{Z}_N)$, with $\mathcal{I}_{T^2}$ a polarization of $H^1(T^2, \mathbb{Z}_N)$. We have $H^1(T^2, \mathbb{Z}_N) = \mathbb{Z}_N^2$, so the conditions of maximality and Dirac quantization in AST map to maximality and isotropy of $\mathcal{I}_{T^2}$.

Examples:

- $\{(1,0),(2,0),\ldots,(N-1,0)\} \leftrightarrow \mathcal{I} = H^2(\mathcal{M}_4, \mathbb{Z}_N)_{\mathsf{NS}}$

  $\mapsto \; SU(N)$
- $\{(0,1),(0,2),\ldots,(0,N-1)\} \leftrightarrow \mathcal{I} = H^2(\mathcal{M}_4, \mathbb{Z}_N)_{\mathsf{RR}}$

  $\mapsto \; (SU(N)/\mathbb{Z}_N)_0$
- $\{a(N,0) + b(0,N)\} \leftrightarrow \mathcal{I} = NH^2(\mathcal{M}_4, \mathbb{Z}_N)_{\mathsf{RR}} + NH^2(\mathcal{M}_4, \mathbb{Z}_N)_{\mathsf{NS}}$

  $\mapsto \; (SU(N^2)/\mathbb{Z}_N)_0$

$SL(2,\mathbb{Z})$ invariance of the partition function on $\mathcal{M}$

In general, the partition function $\sum_{w\in\mathcal{I}} Z_w(\tau)$ is $SL(2,\mathbb{Z})$ invariant for a polarization $\mathcal{I}$ with $\mathcal{P}(w)/2 = 0\ \forall w \in \mathcal{I}$: [Vafa, Witten '94]

$$Z_w(\tau+1) = \omega_N^{\mathcal{P}(w)/2} Z_w(\tau) \tag{30}$$

$$Z_w(-1/\tau) = N^{-b_2/2} \sum_{u\in H^2(\mathcal{M},\mathbb{Z}_N)} \omega_N^{u\cdot w} Z_u(\tau) \tag{31}$$

with $b_2 = \dim H^2(\mathcal{M},\mathbb{R})$ and $\omega_N = \exp(2\pi i/N)$. This holds, since

$$\sum_{w\in\mathcal{I}} \omega_N^{w\cdot u} = \begin{cases} 0 & \text{if } u \notin \mathcal{I} \\ |\mathcal{I}| = N^{b_2/2} & \text{if } u \in \mathcal{I} \end{cases} \tag{32}$$

so

$$\sum_{w\in\mathcal{I}} Z_w(-1/\tau) = N^{-b_2/2} \sum_{u\in H^2(\mathcal{M},\mathbb{Z}_N)} \left(\sum_{w\in\mathcal{I}} \omega_N^{u\cdot w} \right) Z_u(\tau) = \sum_{w\in\mathcal{I}} Z_w(\tau)\,. \tag{33}$$