The Analytic Conformal Bootstrap

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What will this talk be about?

Conformal Field Theories in $D > 2$ dimensions

- Very relevant for Physics:
  - Asymptotic regimes of QFT.
  - Condensed matter/ description of critical points, ...
- Interesting interplay with Mathematics
  - Representation theory, Langlands program,...
- Many important theories are conformal:
  - $4d$ $\mathcal{N} = 4$ SYM.
  - $3d$ Ising model.
  - Critical $O(N)$ models,...
- Ubiquitous in dualities in string and gauge theory.

CFT_D-difficulties

Studying CFT in $D > 2$ is not so easy...

- In general CFTs do not have a Lagrangian description...
- In a Lagrangian theory we can use Feynman diagrams:

$$A(g) = A^{(0)} + gA^{(1)} + \cdots$$

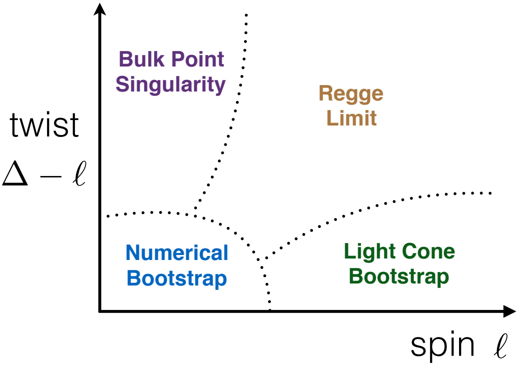
- But generic CFTs don't have a small coupling constant!
- We don't have the luxuries of $D = 2$:
  - No Virasoro Algebra ✗
  - No finite number of primaries ✗
  - Representation theory is less powerful ✗

In spite of all this, progress can be made!

Conformal bootstrap

- Conformal bootstrap: resort to consistency conditions!
  - Conformal symmetry
  - Properties of the OPE
  - Unitarity
  - Crossing symmetry
- This idea was successfully applied to 2d in the eighties! [Ferrara, Gatto, Grillo; Belavin, Polyakov, Zamolodchikov]
- 25 years later it was finally implemented in $D > 2$! [Rattazzi, Rychkov, Tonni, Vichi '08]
- Starting point of an impressive numerical revolution! [Poland's review talk Strings 2015 and still going on]

Today: Analytic results for CFTs from consistency conditions!



Great progress by many people! [L.F.A, Maldacena; Fitzpatrick, Kaplan, Poland, Simmons-Duffin; Komargodski, Zhiboedov; Bissi; Cornalba, Costa, Penedones; Kaviraj, Sen, Sinha; Lukowski, Walters, Wang, Vos, Aharony, Perlmutter, Li, Kulaxizi, Parnachev, , ...]

Other approaches

- Large central charge limit [Heemskerk, Penedones, Polchinski, Sully; Fitzpatrick, Kaplan; L.F.A, Bissi, Lukowski; Rastelli, Zhou,...]
- Solvable protected subsectors [Dolan, Osborn; Beem, Lemos, Liendo, Peelaers, Rastelli, van Rees; Chester, Lee, Pufu, Yacoby, ...]
- Crossing in Mellin space [Gopakumar, Kaviraj, Sen, Sinha; Rastelli, Zhou, L.F.A., Bissi, Lukowski] [Talk by Rastelli]
- Large global charge limit [Hellerman, Orlando, Reffert, Watanabe, Alvarez-Gaume, Loukas, Monin, Pirtskhalava, Rattazzi, Seibold,...]
- ANEC from bootstrap [Maldacena, Shenker, Stanford, Hartman, Jain, Kundu, Hofman, Li, Meltzer, Poland, Rejon-Barrera,...]
- Great progress in 2d [Talks by Kaplan and Yin]

Analytic conformal bootstrap

Our plan is to start with the light-cone bootstrap and expand to all other regions!

Which kind of analytic results will you get today?

Analytic bootstrap

- Results for operators with spin in a generic CFT!

$$\mathcal{O}_{DT} \sim \varphi \partial_{\mu_1} \cdots \partial_{\mu_\ell} \varphi, \qquad \mathcal{O}_{ST} \sim Tr \varphi \partial_{\mu_1} \cdots \partial_{\mu_\ell} \varphi$$

- Study their scaling dimension Δ for large values of the spin ℓ:

$$\Delta(\ell) = \ell + c_0 + \frac{c_1}{\ell} + \frac{c_2}{\ell^2} + \cdots, \quad \text{double-twist}$$

$$\Delta(\ell) = d_0 \log \ell + \ell + d_1 + \frac{c_2}{\ell} + \cdots, \quad \text{single-trace}$$

- We will obtain analytic results to all orders in $1/\ell$ resorting only to consistency conditions.
- Valid for vast families of CFTs!

CFT - Symmetries

Conformal algebra:

- Scale transformations $\rightarrow$ dilatation D
- Poincare Algebra: P_μ and $M_{\mu\nu}$
- Special conformal transformations: K_μ

$$[D, P_\mu] = P_\mu, \quad [D, K_\mu] = -K_\mu, \quad [K_\mu, P_\nu] = \eta_{\mu\nu} D - i M_{\mu\nu}$$

Specific CFTs may have extra symmetries but we will keep the discussion very general.

CFT - Ingredients

Main ingredient:

- Conformal Primary local operators: $\mathcal{O}_{\Delta,\ell}(x)$, $[K_\mu, \mathcal{O}(0)] = 0$

Dimension (Δ) — Lorentz spin (ℓ)

In addition we have descendants $P_{\mu_k}...P_{\mu_1}\mathcal{O}_{\Delta,\ell} = \partial_{\mu_k}...\partial_{\mu_1}\mathcal{O}_{\Delta,\ell}$.

Operators form an algebra (OPE)

$$\mathcal{O}_i(x)\mathcal{O}_j(0) = \sum_{k\in\text{prim.}} C_{ijk}|x|^{\Delta_k-\Delta_i-\Delta_j}\left(\mathcal{O}_k(0)\underbrace{+x^\mu\partial_\mu\mathcal{O}_k(0)+\cdots}_{\text{all fixed}}\right)$$

- CFT data: The set Δ_i and C_{ijk} characterizes the CFT.

CFT - Basics

Main observable:

Correlation functions of primary operators

$$\langle \mathcal{O}_1(x_1)...\mathcal{O}_n(x_n)\rangle$$

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\rangle = \frac{\delta_{ij}}{|x_{12}|^{2\Delta_i}}$$

$$\langle \mathcal{O}_i(1)\mathcal{O}_j(2)\mathcal{O}_k(3)\rangle = \frac{C_{ijk}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{13}|^{\Delta_1+\Delta_3-\Delta_2}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}}$$

Four-point function of identical operators:

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle = \frac{\mathcal{G}(u,v)}{x_{12}^{2\Delta_{\mathcal{O}}}x_{34}^{2\Delta_{\mathcal{O}}}}$$

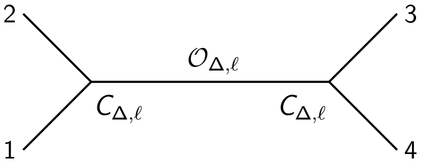
where $u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$, $v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$

Four-point function - properties

Conformal partial wave decomposition

- OPE: $\mathcal{O} \times \mathcal{O} = \sum_i \mathcal{O}_i + \textit{descendants}$

$$\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O} \rangle = \sum_{\Delta,\ell}$$



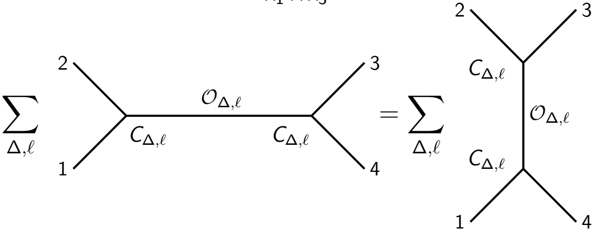
$$\mathcal{G}(u,v) = 1 + \sum_{\Delta,\ell} C^2_{\Delta,\ell} G_{\Delta,\ell}(u,v)$$

Identity operator   Conformal primaries   Conformal blocks

- Conformal blocks: For a given primary, take into account the contribution from all its descendants. Fully fixed function!

Crossing symmetry

$$v^{\Delta_{\mathcal{O}}}\mathcal{G}(u,v) \underbrace{=}_{x_1 \leftrightarrow x_3} u^{\Delta_{\mathcal{O}}}\mathcal{G}(v,u)$$

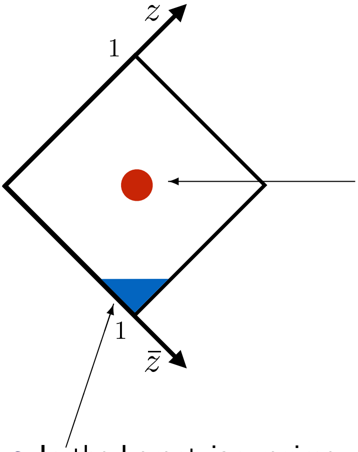


A remarkable...but hard equation!

$$\underbrace{v^{\Delta_{\mathcal{O}}}\left(1+\sum_{\Delta,\ell} C^2_{\Delta,\ell} G_{\Delta,\ell}(u,v)\right)}_{\text{Easy to expand around } u=0,\, v=1} = \underbrace{u^{\Delta_{\mathcal{O}}}\left(1+\sum_{\Delta,\ell} C^2_{\Delta,\ell} G_{\Delta,\ell}(v,u)\right)}_{\text{Easy to expand around } u=1,\, v=0}$$

Numerical vs Analytic bootstrap

Study this equation in different regions, $u = z\bar{z}, v = (1-z)(1-\bar{z})$



- In the Euclidean regime $\bar{z} = z^*$.
- We can study crossing around $u = v = \frac{1}{4}$
- Starting point of the numerical bootstrap.

- In the Lorentzian regime $z, \bar{z}$ are independent real variables and we can consider $u, v \to 0$.
- Starting point of the analytic (light-cone) bootstrap!

Analytic bootstrap

- Why is this a good idea?

$$v^{\Delta_{\mathcal{O}}}\left(1+\sum_{\Delta,\ell} C^2_{\Delta,\ell} G_{\Delta,\ell}(u,v)\right) \quad = \quad u^{\Delta_{\mathcal{O}}}\left(1+\sum_{\Delta,\ell} C^2_{\Delta,\ell} G_{\Delta,\ell}(v,u)\right)$$

Direct channel $\Leftrightarrow$ Crossed channel

- Very complicated interplay between l.h.s. and r.h.s. ... but:

Operators with large spin

DT operators with large spin $\Leftrightarrow$ Identity operator
ST operators with large spin $\Leftrightarrow$ ST operators with large spin

Conformal blocks - technicalities

- Eigenfunctions of a Casimir operator

$$\mathcal{C}\, G_{\Delta,\ell}(u,v) = J^2 G_{\Delta,\ell}(u,v)$$

where $J^2 = (\ell+\Delta)(\ell+\Delta-1) \sim \ell^2$

- Small u limit:

$$G_{\Delta,\ell}(u,v) \sim u^{\tau/2} f^{coll}_{\tau,\ell}(v), \quad \tau = \Delta - \ell$$

We will introduce the notation

$$G_{\Delta,\ell}(u,v) \equiv u^{\tau/2} f_{\tau,\ell}(u,v)$$

- Small v limit:

$$f_{\tau,\ell}(u,v) \sim \log v$$

Overlays

- Consider the $v \ll 1$ limit of the crossing equation: $C^2_{\Delta,\ell} \to a_{\tau,\ell}$

$$v^{\Delta_{\mathcal{O}}}\left(1+\sum_{\tau,\ell} a_{\tau,\ell} u^{\tau/2} f_{\tau,\ell}(u,v)\right) = u^{\Delta_{\mathcal{O}}}\left(1+\sum_{\tau,\ell} a_{\tau,\ell} v^{\tau/2} f_{\tau,\ell}(v,u)\right)$$

$$\Downarrow$$

$$1+\sum_{\tau,\ell} a_{\tau,\ell} u^{\tau/2} f_{\tau,\ell}(u,v) = \frac{u^{\Delta_{\mathcal{O}}}}{v^{\Delta_{\mathcal{O}}}} + \underbrace{\text{subleading terms}}_{\text{rest of operators sorted by twist}}$$

- The r.h.s. is divergent as $v \to 0$.
- Each term on the l.h.s. diverges as $f_{\tau,\ell}(u,v) \sim \log v$.
- In order to reproduce the divergence on the right, we need infinite operators, with large spin and whose twist approaches $\tau = 2\Delta_{\mathcal{O}}$ (actually $\tau_n = 2\Delta_{\mathcal{O}} + 2n$)

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Example: Generalised free fields

- Simplest solution: Large N CFTs - Generalised free fields

$$\mathcal{G}^{(0)}(u,v) = 1 + \left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}} + u^{\Delta_{\mathcal{O}}}$$

- Intermediate ops: Double twist operators: $\mathcal{O}\Box^n \partial_{\mu_1} \cdots \partial_{\mu_\ell} \mathcal{O}$

$$\begin{aligned} \tau_{n,\ell} &= 2\Delta_{\mathcal{O}} + 2n \\ a_{n,\ell} &= a^{(0)}_{n,\ell} \end{aligned}$$

- Their OPE coefficients are such that the divergence of a single conformal block $(\sim \log v)$, as $v \to 0$, is enhanced!

$$1 + \sum_{\tau,\ell} a^{(0)}_{\tau_n,\ell} u^{\tau_n/2} f_{\tau_n,\ell}(u,v) = 1 + \underbrace{\left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}}} + u^{\Delta_{\mathcal{O}}}$$

But this divergence is quite universal!

Large spin sector

Additivity property [Fitzpatrick, Kaplan, Poland, Simmons-Duffin; Komargodski, Zhiboedov]

In any CFT with $\mathcal{O}$ in the spectrum, crossing symmetry implies the existence of double twist operators with arbitrarily large spin and

$$\tau_{n,\ell} = 2\Delta_{\mathcal{O}} + 2n + \mathcal{O}\left(\frac{1}{\ell}\right)$$

$$a_{n,\ell} = a^{(0)}_{n,\ell}\left(1 + \mathcal{O}\left(\frac{1}{\ell}\right)\right)$$

- All CFTs have a large spin sector, for which the operators become "free"!
- Can we do perturbations around large spin? YES!

General picture

- We would like to exploit the following idea

$$\sum_{\tau,\ell} a_{\tau,\ell} u^{\tau/2} f_{\tau,\ell}(u,v) \quad = \quad \frac{u^{\Delta_{\mathcal{O}}}}{v^{\Delta_{\mathcal{O}}}} + \cdots$$

Behaviour at large spin $\Leftrightarrow$ Enhanced divergences as $v \to 0$

- The presence of the identity on the r.h.s. already led to a remarkable result!
- Let's take this to the next level!
- We will focus in corrections to GFF.

New ingredient: Twist conformal blocks

- GFF has accumulation points in the twist at

$$\tau_n = 2\Delta_{\mathcal{O}} + 2n$$

- It is convenient to define "twist" conformal blocks, the contribution to the 4pt function from a given twist.

$$\sum_{\ell} a^{(0)}_{\tau,\ell} u^{\tau/2} f_{\tau,\ell}(u,v) \equiv H_\tau(u,v)$$

- And twist conformal blocks with insertions:

$$\sum_{\ell} \frac{a^{(0)}_{\tau,\ell}}{J^{2m}} u^{\tau/2} f_{\tau,\ell}(u,v) \equiv H^{(m)}_\tau(u,v)$$

Twist conformal blocks

Properties

- Correlator decomposition

$$\mathcal{G}^{(0)}(u,v) = \sum_n H^{(0)}_{\tau_n}(u,v)$$

- Recurrence relation

$$\mathcal{C} H^{(m+1)}_{\tau}(u,v) = H^{(m)}_{\tau}(u,v)$$

- Prescribed behaviour at small u and small v.

$$\begin{aligned} H^{(m)}_{\tau}(u,v) &\sim u^{\tau/2}, \qquad \text{small } u \\ H^{(m)}_{\tau}(u,v) &\sim v^{-\Delta_{\mathcal{O}}+m}, \qquad \text{small } v \end{aligned}$$

The functions $H^{(m)}_{\tau}(u,v)$ can be systematically constructed!

Corrections to GFF

- Let us compute $1/N$ corrections to large N CFTs/GFF!

$$\mathcal{G}(u,v) = \mathcal{G}^{(0)}(u,v) + \frac{1}{N^2}\mathcal{G}^{(1)}(u,v) + \cdots$$

Two Sources of corrections

1. Double twist operators will acquire corrections:

$$\begin{aligned} \tau_{n,\ell} &= 2\Delta_{\mathcal{O}} + 2n + \frac{1}{N^2}\gamma_{n,\ell} + \cdots \\ a_{n,\ell} &= a^{(0)}_{n,\ell} + \frac{1}{N^2}a^{(1)}_{n,\ell} + \cdots \end{aligned}$$

2. We can also have new intermediate operators at order $1/N^2$.

Which corrections are consistent with crossing symmetry

$$\mathcal{G}^{(1)}(u,v) = \left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}} \mathcal{G}^{(1)}(v,u)$$

Corrections to GFF

- Conformal partial wave expansion

$$\mathcal{G}(u,v) = \sum_{\tau,\ell} a_{\tau,\ell} u^{\tau/2} f_{\tau,\ell}(u,v)$$

- Plug $\tau_{n,\ell} = 2\Delta_{\mathcal{O}} + 2n + \frac{1}{N^2}\gamma_{n,\ell} + \cdots$ and expand in $\frac{1}{N^2}$:

$$\mathcal{G}^{(1)}(u,v) = \sum_{n,\ell} a^{(0)}_{\tau,\ell} u^{\Delta_{\mathcal{O}}+n} \frac{\gamma_{n,\ell}}{2} f_{\tau,\ell}(u,v) \log u + \cdots$$

- Assume $\gamma_{n,\ell}$ admits an expansion in $1/J$:

$$\gamma_{n,\ell} = 2\frac{b_{n,1}}{J^2} + 2\frac{b_{n,2}}{J^4} + \cdots$$

$$\Downarrow$$

$$\mathcal{G}^{(1)}(u,v) = \sum_{n,m} b_{n,m} H^{(m)}_{\tau_n}(u,v) \log u + \cdots$$

Corrections to GFF

Plugging this into the crossing equation...

$$\sum_{n,m} b_{n,m} H_n^{(m)}(u,v) = \left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}} \mathcal{G}^{(1)}(v,u)\Big|_{\log u}$$

- Now both sides can be expanded around small v!
- Matching divergences we can fix all coefficients $b_{n,m}$. Hence $\gamma_{n,\ell}$ to all orders in $1/\ell$!
- Crossing equation has become algebraic!

Corrections to GFF

$$\sum_{n,m} b_{n,m} H_n^{(m)}(u,v) = \left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}} \mathcal{G}^{(1)}(v,u)\Big|_{\log u}$$

Case 1: No new operators at order $1/N^2$

- The r.h.s has no divergences as $v \to 0$.
- All $H_n^{(m)}(u,v)$ on the l.h.s. must be absent.
- $\gamma_{n,\ell}$ vanishes to all orders in $1/\ell$!

While truncated in the spin are allowed! [Heemskerk, Penedones, Polchinski, Sully]

Simplest solution, $d = 2, \Delta = 2$

$$\gamma_{n,\ell} = \begin{cases} \frac{3}{3+2n} & \ell = 0 \\ 0 & \ell \neq 0 \end{cases}$$

Corrections to GFF

Case 2: New single-trace operators at order $1/N^2$, *e.g.* $\mathcal{O}$ itself:

$$\mathcal{O} \times \mathcal{O} = 1 + [\mathcal{O}, \mathcal{O}]_{n,\ell} + \frac{1}{N^2}\mathcal{O}$$

- Now the situation is more interesting:

$$\sum_{n,m} b_{n,m} H^{(m)}_{\tau_n}(u,v) \sim \left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}} v^{\Delta_{\mathcal{O}}/2} f_{\mathcal{O}}(v,u)\Big|_{\log u}$$

- Reproducing the divergences as $v \to 0$ fixes all $b_{n,m}$, and hence $\gamma_{n,\ell}$ to all orders in $1/\ell$!

e.g. $\Delta = 2$

$$\gamma_{n,\ell} = -\frac{2}{(\ell+n+2)(\ell+n+1)}$$

- This result was not known for generic Δ!

Wider perspective on CFT

The additivity property we have seen is a particular example of a more general one

If two operators $\mathcal{O}_{\tau_1}, \mathcal{O}_{\tau_2}$ of twists τ_1 and τ_2 are part of the spectrum then there is a tower of operators $[\mathcal{O}_{\tau_1}, \mathcal{O}_{\tau_2}]_{n,\ell}$ of twist

$$\tau_{[\mathcal{O}_{\tau_1},\mathcal{O}_{\tau_2}]_{n,\ell}} = \tau_1 + \tau_2 + 2n + \mathcal{O}\left(\frac{1}{\ell}\right)$$

- This additivity in the twist property should make you happy and sad at the same time!

The spectrum of generic CFTs is hard!

- $\mathcal{O}$ is part of the spectrum.
- $[\mathcal{O}, \mathcal{O}]_{n,\ell}$ is also part of the spectrum.
- And $[[\mathcal{O}, \mathcal{O}]_{n_1,\ell_1}, [\mathcal{O}, \mathcal{O}]_{n_2,\ell_2}]_{n_3,\ell_3}$ too, and so on!

Large spin expansions for non-perturbative CFTs

In non-perturbative CFTs the spectrum is very rich. Hard to apply our idea

$$\sum_{\tau,\ell} a_{\tau,\ell} u^{\tau/2} f_{\tau,\ell}(u,v) \quad = \quad \text{Rich spectrum in the crossed channel}$$
$$\text{Behaviour at large spin} \quad \Leftrightarrow \quad \text{complicated divergences as } v \to 0$$

However

$$\gamma_\ell = -\frac{c_0}{\ell^{\tau_\epsilon}} + \cdots \Leftrightarrow \text{Operator } \mathcal{O}_\epsilon \text{ in the crossed channel}$$

- Having control on the low twist spectrum of the theory, *e.g.* through the numerical bootstrap, we can compute the leading terms in the $1/\ell$ expansion of the anomalous dimensions of DT operators.

3d Ising Model

- Spin operator σ: $\Delta_\sigma = 0.518151 = 1/2 + \gamma_\sigma$.
- Higher spin-operators $\sigma \partial_{\mu_1} \cdots \partial_{\mu_\ell} \sigma$, $\quad \Delta_\ell = 1 + \gamma_\ell$

One predicts [Work with Zhiboedov]

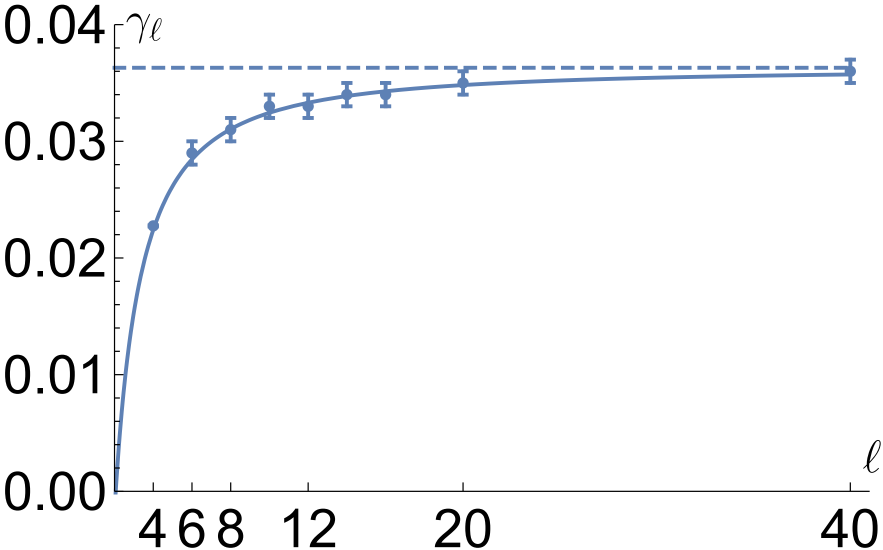
$$\gamma_\ell \sim 2\gamma_\sigma - \frac{0.0027}{\ell} - \frac{0.0926}{\ell^{1.4126}} - \frac{0.0024}{\ell^{2.4126}} + \cdots$$

- Next corrections are hard to control but small for large spin.
- $\ell = 4$ is already large!

Two important comments

- Powerful interplay with the numerical bootstrap, taken to the next level by [Simmons-Duffin]!
- The convergence properties of these series was stablished in [Caron-Huot] where it was also understood why they work so well!

3d Ising - Comparison with numerical results



Large spin perturbation theory

- If the CFT has a small parameter we are better of, as this parameter further organises the problem.

The range of applicability of our method is very vast [Work with Aharony, Bissi, Perlmutter, Zhiboedov]

- ▷ Weakly coupled gauge theories.
- ▷ Theories with weakly broken higher spin symmetry.
- ▷ $1/N^4$ corrections to GFF.

Each one interesting on its own right! The series in $1/\ell$ can be resummed in all examples!

Weakly coupled conformal gauge theories

Single trace operators with spin

$$\mathcal{O}_\ell \sim Tr\varphi\partial_{\mu_1}\cdots\partial_{\mu_\ell}\varphi$$

- Highly non-trivial fact:

$$\Delta - \ell = f(g)\log\ell + \cdots$$

Get this from crossing

- Study the following correlator in a weakly coupled gauge theory

$$\langle\mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle, \quad \mathcal{O} = Tr\varphi^2$$

- The $\mathcal{O}_\ell$ are the operators with leading twist in the theory

$$\tau_\ell = \Delta_\ell - \ell = \Delta_{\mathcal{O}} + \gamma_\ell \leftarrow \text{Small in pert. theory}$$

- Focus on their contribution by taking the small u, v limit!

Weakly coupled conformal gauge theories

$$u^{-\Delta_{\mathcal{O}}} \sum_{\ell} a_\ell u^{\Delta_{\mathcal{O}}/2+\gamma_\ell/2} f^{\ell}_{coll}(v) = v^{-\Delta_{\mathcal{O}}} \sum_{\ell} a_\ell v^{\Delta_{\mathcal{O}}/2+\gamma_\ell/2} f^{\ell}_{coll}(u)$$

- ST operators with large spin map to themselves!
- Match divergences on both sides $\rightarrow$ constraints on γ_ℓ

Logarithmic behaviour?

- General ansatz: $\gamma_\ell = f \log \ell + a_2 \log^2 \ell + a_3 \log^3 \ell + \cdots$
- Crossing symmetry: $\gamma_\ell = f \log \ell + \cancel{a_2 \log^2 \ell} + \cancel{a_3 \log^3 \ell} + \cdots$

Higher order corrections? At one loop crossing is very powerful!

$$\gamma_\ell = g \log \ell + \underbrace{c_0 + \frac{c_1}{\ell} + \frac{c_2}{\ell^2} + \frac{c_3}{\ell^3} + \cdots}_{\text{all fixed!}} \rightarrow g\, S_1(\ell)$$

CFT with weakly broken HS symmetry

CFT with HS symmetry

- The following is part of the spectrum
  - Fundamental scalar field φ

$$\partial_\mu \partial^\mu \varphi = 0 \to \Delta_\varphi = \frac{d-2}{2}$$

  - Tower of HS conserved currents $J^{(s)} = \varphi \partial_{\mu_1} \cdots \partial_{\mu_s} \varphi$,

$$\text{conservation} \to \Delta_s = d - 2 + s$$

Weakly broken HS symmetry

- The fundamental field φ and HS currents $J^{(s)}$ acquire an anomalous dimensions:

$$\begin{aligned} \Delta_\varphi &= \frac{d-2}{2} + g\gamma_\varphi + \cdots \\ \Delta_s &= d - 2 + s + g\gamma_s + \cdots \end{aligned}$$

Which anomalous dimensions are consistent with crossing symmetry?

CFT with weakly broken HS symmetry

Crossing for the correlator $\langle \varphi\varphi\varphi\varphi \rangle$ fixes the spectrum!

e.g. Single scalar in $d = 4$

$$\gamma_{\varphi^2} = g + \cdots \,, \quad \gamma_\varphi = \frac{1}{12} g^2 + \cdots \,, \quad \gamma_\ell = 2\gamma_\varphi \left(1 - \frac{6}{\ell(\ell+1)} \right) + \cdots$$

We studied several models. Again crossing symmetry fixes the spectrum!

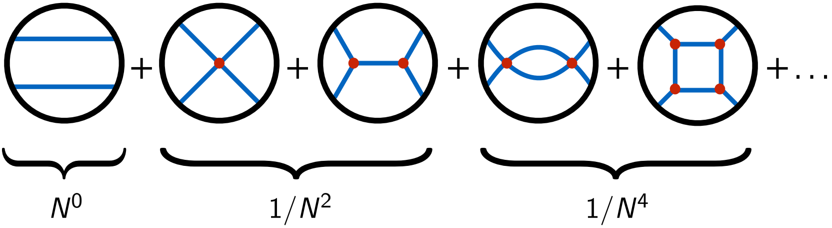
- Wilson-Fisher models around $d = 4$ ✓
- Large N critical $O(N)$ models in $2 < d < 4$ ✓
- Cubic models around $d = 6$ ✓

AdS/CFT

Large N CFT in D-dimensions (GFF + corrections)	$\Leftrightarrow$	Gravitational theory in AdS_{D+1}

$\frac{1}{N^2}$ expansion in CFT $\leftrightarrow$ loops in AdS/powers of G_N.

$\mathcal{G} =$



- Diagrams in AdS are hard to compute...Use crossing for the CFT!

$1/N^4$ corrections to GFF

- Loops in *AdS* are a largely unexplored subject.
- We can approach this by studying $1/N^4$ corrections to GFF!

$$\tau_{n,\ell} = 2\Delta_{\mathcal{O}} + 2n + \frac{1}{N^2}\gamma^{(1)}_{n,\ell} + \frac{1}{N^4}\gamma^{(2)}_{n,\ell} + \cdots$$

- Consistency conditions fix $\gamma^{(2)}_{n,\ell}$ from the leading order solution!
- Even $1/N^4$ corrections to $\mathcal{N} = 4$ SYM at strong coupling! equivalent to loop corrections to supergravity on $AdS_5 \times S^5$.

$$\Delta_{0,2} = 6 - \frac{4}{N^2} - \frac{45}{N^4} + \cdots$$

- First non-protected quantity ever computed to this order in *AdS*/*CFT*!

Conclusions

- Generic CFTs have a large spin sector which becomes essentially free and we have shown how to perform a perturbation around that sector.
- The method applies to vast families of CFTs.
- It provides an alternative to Feynman diagram computations.
- It provides an on-shell gauge invariant way to study weakly coupled gauge theories.
- It allows to learn about loops in AdS and quantum gravity.
- It allows connections with the numerical bootstrap.

This is just the beginning!!